Assignment # 9 Solutions

1 a) Consider $\underline{v}^T A \underline{v} = \underline{v}^T U^T U \underline{v} = (U\underline{v})^T U\underline{v}$

suppose we let $U\underline{v} = \underline{w}$ and note that $\underline{w} = 0$ iff $\underline{v} = \underline{0}$ as U is invertible

Then $\underline{v}^T A \underline{v} = \underline{w}^T \underline{w} = \sum_{i=1}^{n} w_i^2 > 0$ (since $\underline{w} \in \mathbb{R}^n$, $\underline{w} \neq \underline{0}$)

We conclude that A is + def.

b) We know that A is + def iff both the eigenvalues of A are positive
consider the char. poly. of A, $\Delta_A(t) = t^2 - (a+c)t + ac - b^2$
If λ_1 and λ_2 are the roots then $\lambda_1 + \lambda_2 = a+c$ $\lambda_1\lambda_2 = ac - b^2$.

A is + def iff $\lambda_1 > 0$ and $\lambda_2 > 0$ iff $\lambda_1 + \lambda_2 > 0$ and $\lambda_1 \lambda_2 > 0$
iff $(a+c) > 0$ and $ac - b^2 > 0$
iff $(a+c) > 0$ and $ac > b^2$

The second condition in this last expression is telling us that a and c have the same sign and the first condition is telling us that this sign is positive. This second condition is identical to the statement "$\det(A) > 0$".

Thus we conclude that A is + def iff $a > 0$ and $\det(A) > 0$.

2 a) A is symmetric so there exists an orthogonal matrix P $\ni$ $P^{-1}AP = \text{diag}(\lambda_1, \dots, \lambda_n) = D$, where λ_i are the eigenvalues of A. Also recall that A is invertible so that none of the eigenvalues are zero.

Then $A = PDP^{-1}$ and $A^2 = PD^2P^{-1} \iff P^{-1}A^2P = D^2$.
Note that $D^2 = \text{diag}(\lambda_1^2, \dots, \lambda_n^2)$ and each $\lambda_i^2 > 0$.

We conclude that the eigenvalues of A^2 are all greater than zero and thus A^2 is positive definite.

b) $A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ $\quad A^2 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$

Since $13 > 0$ and $\det(A^2) = 65 - 64 = 1 > 0$
we conclude from Q 1b) that A^2 is positive definite.

As for A, $\det(A) = 3 - 4 = -1 < 0$ which means that the two eigenvalues of A have different signs. A is not positive definite.

c) A is + def $\iff$ A has n +tive eigenvalues, $\lambda_1, \dots, \lambda_n$
The eigenvalues of A^k are λ_i^k and these must be +tive.

We conclude that A^k is + def.

d) A is + def and has eigenvalues $\lambda_1, \dots, \lambda_n$.
The eigenvalues of A^k are λ_i^k and these are +tive as A^k is + def.
Since k is odd we can conclude that $\lambda_i = (\lambda_i^k)^{1/k} > 0$.
Thus A is + def.

Q3 a) $\quad Q(\underline{x}) = 9x^2 - 8xy + 3y^2 = \underline{x}^T A \underline{x}$ where $A = \begin{pmatrix} 9 & -4 \\ -4 & 3 \end{pmatrix}$

The char poly of A is $\Delta_A(t) = (9-t)(3-t) - 16 = (t-1)(t-11)$.
Thus the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 11$

The corresponding eigenvectors are $\underline{v}_1 : (A - I)\underline{v}_1 = \underline{0}$

$\Leftrightarrow \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \underline{v}_1 = \underline{0}$ a choice is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

and $\underline{v}_2 : (A - 11I)\underline{v}_2 = \underline{0} \Leftrightarrow \begin{pmatrix} 2 & -4 \\ -4 & -8 \end{pmatrix} \underline{v}_2 = \underline{0} \Leftrightarrow \underline{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
is a choice

n.b. you could use the fact that $\underline{v}_1 \cdot \underline{v}_2 = 0$ to find $\underline{v}_2$.

Since λ_1 and λ_2 are positive then $Q(\underline{x}) = c$ is an ellipse and its principal axes are straight lines through the origin parallel to $\underline{v}_1$ and $\underline{v}_2$, that is $y = 2x$ and $y = -\frac{1}{2}x$.

The max and min of Q on $\|\underline{x}\| = 1$ are 11 and 1 respectively.

Q3 b) $\quad Q(\underline{x}) = 8x^2 + 6xy = \underline{x}^T A \underline{x}$ where $A = \begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix}$

The char poly of A is $\Delta_A(t) = (8-t)(-t) - 9 = (t-9)(t+1)$
and so the eigenvalues are 9 and -1.

The corresponding eigenvectors are

$\underline{v}_1$: $\quad (A - 9I)\underline{v}_1 = \underline{0} \iff \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \underline{v}_1 = \underline{0}$, a choice is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\underline{v}_2$: $\quad (A + I)\underline{v}_2 = \underline{0} \iff \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \underline{v}_2 = 0$, a choice is $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Since λ_1 and λ_2 have different signs, then $Q(\underline{x}) = c$ is an hyperbola and its principal axes are $y = \frac{1}{3}x$ and $y = -3x$.

The max and min of Q on $\|\underline{x}\| = 1$ are 9 and -1 respectively.

Q4 Consider $Q(\underline{x}) = 2x^2 - 72xy + 23y^2 = \underline{x}^T A \underline{x}$, $A = \begin{pmatrix} 2 & -36 \\ -36 & 23 \end{pmatrix}$

$$\Delta_A(t) = (2-t)(23-t) - (36)^2 = t^2 - 25t - 1250$$
$$= (t-50)(t+25)$$

Let $\lambda_1 = 50$ and $\lambda_2 = -25$

$(A - 50I)\,\underline{v}_1 = \underline{0} \iff \begin{pmatrix} -48 & -36 \\ -36 & -27 \end{pmatrix} \underline{v}_1 = \underline{0}$ a choice is $\underline{v}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

We choose an orthogonal vector for $\underline{v}_2$, $\underline{v}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

We now normalise these vectors to produce an orthogonal matrix:

Let $P = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$ then $P^{-1} A P = \text{diag}(50, -25)$

If we let $\begin{pmatrix} u \\ v \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = P^T \begin{pmatrix} x \\ y \end{pmatrix}$ (*) then $Q(\underline{x}) = -50$ becomes

$$50u^2 - 25v^2 = -50 \iff -u^2 + v^2/2 = 1$$

We first sketch this hyperbola in the u-v plane

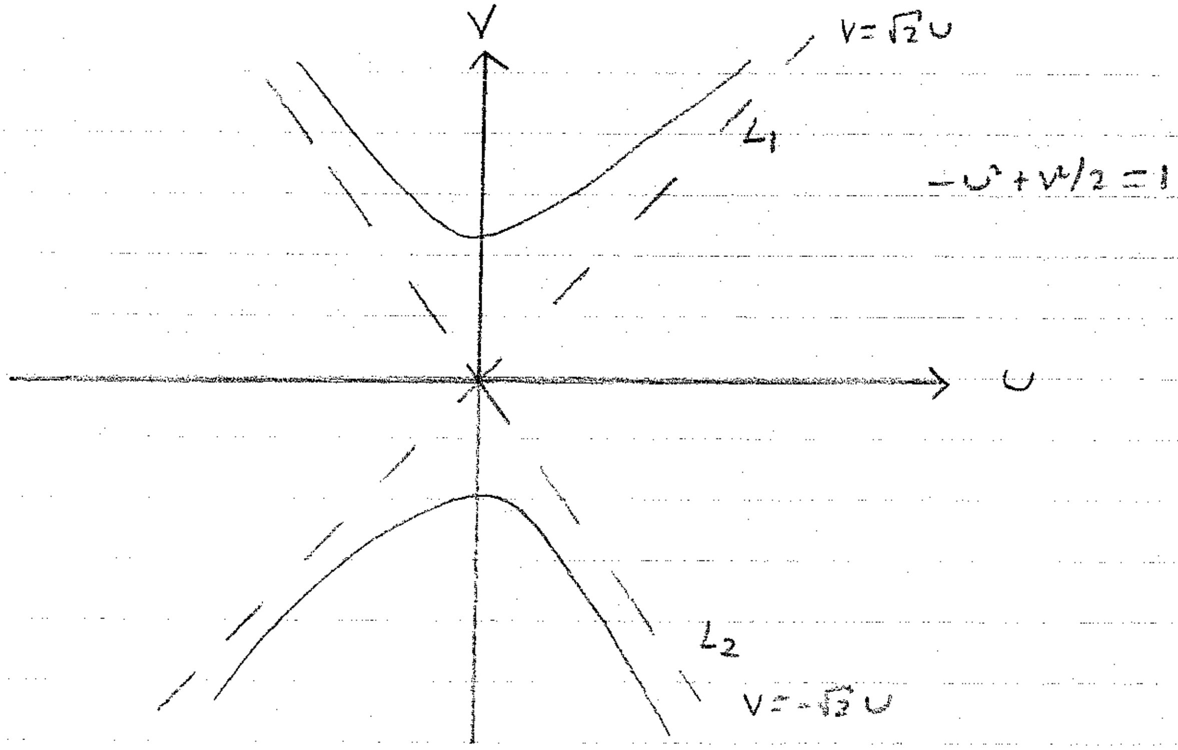
$$v^2/2 - u^2 = 1 \iff (v - \sqrt{2}\,u)(v + \sqrt{2}\,u) = 2$$

and this hyperbola has asymptotes given by $v = \pm\sqrt{2}\,u$.

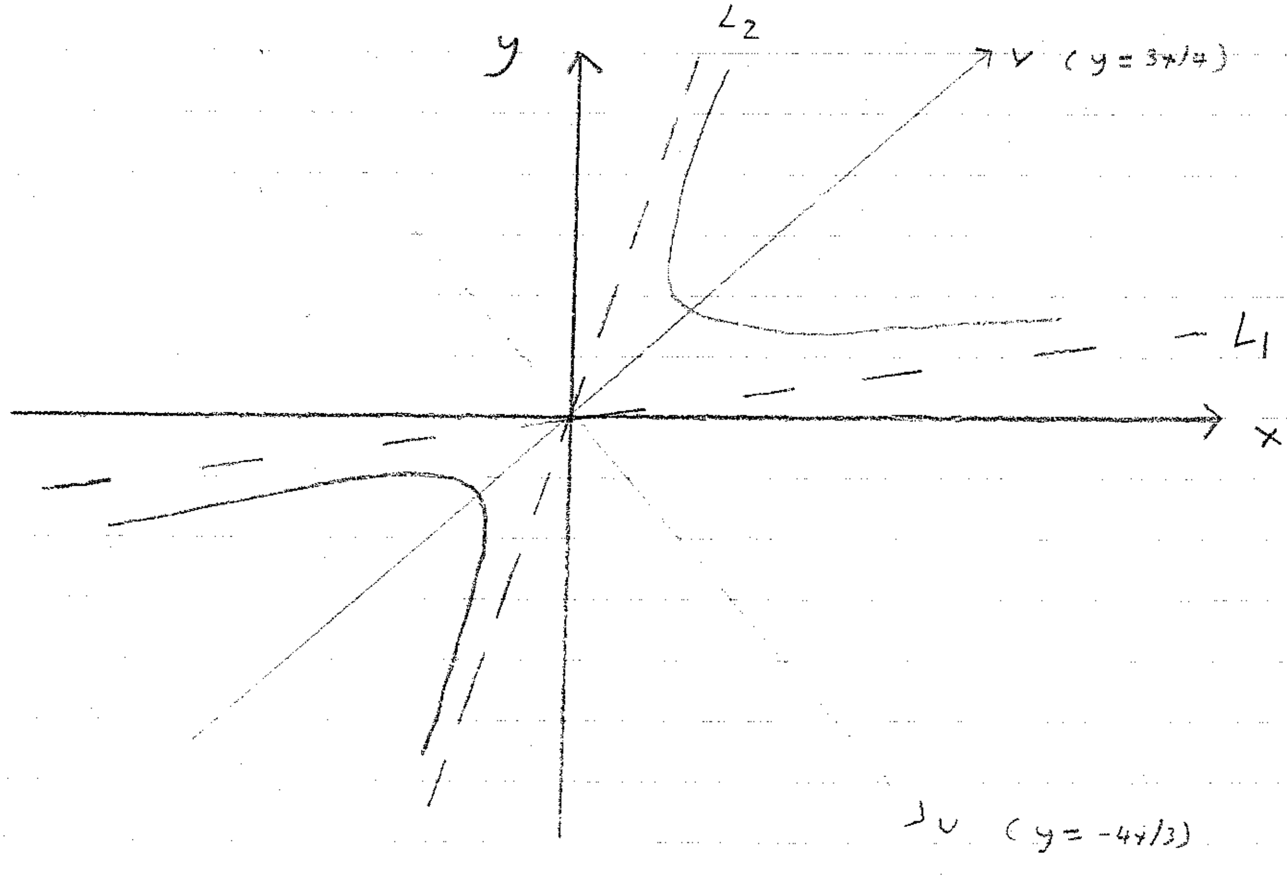
Note that in the x-y plane, the line $v = \alpha u$ becomes (using (*))

$$4x + 3y = \alpha(3x - 4y) \iff y = \left(\frac{3\alpha - 4}{3 + 4\alpha}\right) x$$

and thus the asymptotes become $y = \left(\frac{3\sqrt{2} - 4}{3 + 4\sqrt{2}}\right) x$, $y = \left(\frac{-3\sqrt{2} - 4}{3 - 4\sqrt{2}}\right) x$



Sketch of the conic $Q(\vec{x}) = -50$



Q5 $\quad Q(\underline{x}) = \underline{x}^T A \underline{x} \qquad A = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

$$\Delta_A(t) = (-2-t)\left[(-1-t)(-t)-4\right] - 2\left[2(-t)-2.0\right]$$
$$= -(2+t)(t^2+t-4)+4t = -2t^2-2t+8-t^3-t^2+4t+4t$$
$$= -t^3-3t^2+6t+8 = (t+4)(t-2)(t+1)$$

The eigenvalues of A are $-4, -1$ and 2.
Thus the max of $Q(\underline{x})$ on $\|\underline{x}\|=1$ is 2

We now need the corresponding eigenvector : $(A-2I)\underline{v} = \underline{0}$ is

$$\begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix} \underline{v} = \underline{0} \iff \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \underline{v} = \underline{0} \qquad \text{a soln. is} \quad \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

The normalized eigenvector is $\pm \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

The two unit vectors $\underline{x}$ in $\mathbb{R}^3$ at which $Q(\underline{x})$ is maximized on $\|\underline{x}\|=1$ are $\pm \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

n.b. the question only asks for one of these.

Q6 In each case we need to solve the eigenvalue problem for A^TA (B^TB) and then produce the required orthogonal matrices.

a) $A = \begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}$ $M = A^TA = \begin{pmatrix} 6 & -7 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix} = \begin{pmatrix} 85 & -30 \\ -30 & 40 \end{pmatrix}$

The char. poly. is $\Delta_A(t) = (t-100)(t-25)$. Let $\lambda_1 = 100$ $\lambda_2 = 25$

$\sigma_1 = 10$ $\sigma_2 = 5$

$(M - 100I)\,\underline{v} = \underline{0}$ is $\begin{pmatrix} -15 & -30 \\ -30 & -60 \end{pmatrix}\underline{v} = \underline{0}$ a soln is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

let $\underline{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$(M - 25I)\,\underline{v} = 0$ is $\begin{pmatrix} 60 & -30 \\ -30 & 15 \end{pmatrix}\underline{v} = \underline{0}$ a soln is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

let $\underline{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$V = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

We now compute $\frac{1}{\sigma_i} A\underline{v}_i$

$$\frac{1}{\sigma_1} A\underline{v}_1 = \frac{1}{10}\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\frac{1}{\sigma_2} A\underline{v}_2 = \frac{1}{5}\begin{pmatrix} 6 & 2 \\ -7 & 6 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$U = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$ $S = \operatorname{diag}(10, 5)$.

$A = USV^T$ is (an) SVD.

b) $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ $M = B^T B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

The char. poly. is $(2-t)^2 - 1 = (3-t)(1-t)$ $\lambda_1 = 3$ $\lambda_2 = 1$
$\sigma_1 = \sqrt{3}$ $\sigma_2 = 1$

$(M - 3I)\underline{v} = \underline{0}$ is $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\underline{v} = \underline{0}$ a soln is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$(M - 1I)\underline{v} = \underline{0}$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\underline{v} = \underline{0}$ a soln is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\underline{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

We now compute $\frac{1}{\sigma_i} B\underline{v}_i$

$$\frac{1}{\sigma_1} B\underline{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\frac{1}{\sigma_2} B\underline{v}_2 = \frac{1}{1}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The remaining column of U is obtained by producing a vector orthogonal to these two and then scaling it.
A soln by inspection is $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Let $U = \begin{pmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$ then $B = USV^T$
is (an) SVD.