C&O 367/CM442
Assignment 1

Due on Thursday, Jan. 12, 2006 Instructor H. Wolkowicz

Notes:

1. Please explain your answers and proofs carefully. A yes or no does not constitute a valid answer; nor does a numerical value with no explanation constitute a valid answer.

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1. GEOMETRY

For a vector $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, we define the *(Euclidean) norm* and *(Euclidean) inner product*

$$||x|| := (x_1^2 + \cdots + x_n^2)^{1/2},$$

$$\langle x, y \rangle := x \cdot y := x_1 y_1 + \cdots + x_n y_n.$$

Theorem 1 *(Cauchy-Schwarz)* $|x \cdot y| \leq ||x|| ||y||$, *with equality if and only if* $x = \lambda y$, *for some* λ.

EXERCISES

(a) (6) Explain the geometrical significance for the vectors x and y of:

i.

$$x \cdot y = 0.$$

ii.

$$x \cdot y > 0.$$

iii.

$$x \cdot y = 1, \quad \|x\| = \|y\| = 1.$$

(b) (4) Prove that the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = a \cdot x + \alpha$ is continuous, where a is a given vector and α is a given scalar.

2. CALCULUS

Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}$. Then the *gradient* is denoted

$$\nabla g := \begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \cdots \\ \frac{\partial g}{\partial x_n} \end{pmatrix}$$

EXERCISES

(a) (2) If $g(x) = ||x||$, calculate $\nabla g(x)$ for $x \neq 0$.

(b) (4) Suppose $f : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}^n$, and $f(t) := g(a - tb)$. Calculate $f'(t)$.

3. TAYLOR SERIES

Suppose $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable at $x = a$. The *Taylor Series* of f about a is:

$$f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \frac{1}{3!} f'''(a)(x - a)^3 + \cdots$$

EXERCISES
Write down the Taylor series of:

(a) (2)
$$f(x) = x^3, \text{ about } x = 1.$$

(b) (2)
$$f(x) = \log(1 + x), \text{ about } x = 0.$$

4. TOPOLOGY

The *open ball* $B(x; r) := \{y \in \mathbb{R}^n : ||x - y|| < r\}$. Suppose that D is a subset of $\mathbb{R}^n$.

Interior: $x \in \operatorname{int} D$ if there exists $r > 0$ with $B(x; r) \subset D$.

Closure: $x \in \operatorname{cl} D$ if there exists a sequence $x^k \in D$ with $x^k \to x$.

Boundary: $x \in \partial D$ if $x \in \operatorname{cl} D \setminus \operatorname{int} D$.

D is *open* if $D = \operatorname{int} D$. D is *closed* if $D = \operatorname{cl} D$.

EXERCISES

(a) (6) For each of the following sets, find the interior, the closure, and the boundary. Then determine which of the sets are open, closed, neither, or both.

i.

$$\left\{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\right\}.$$

ii.

$$\left\{(x_1, x_2) : x_1 > 0, x_2 > 0\right\}.$$

iii.

$$\left\{(x_1, x_2) : x_1 > 0, x_2 \geq 0\right\}.$$

iv.

$$\mathbb{R}^n$$

v.

$$\left\{(x_1, x_2) : x_1^2 + x_2^2 < 0\right\}.$$

vi.

$$\emptyset.$$

(b) i. (4) Prove that D is closed if and only if the complement D^c is open.

ii. (4) Prove that $x \in \partial D$ if and only if for any $r > 0$ there exists a $y \in B(x; r) \cap D$ and a $z \in B(x; r) \cap D^c$.

5. MATRICES

EXERCISES

(a) (4) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

i. Calculate the determinant of A.

ii. Calculate the rank of A.

iii. What is the rank of A^T?

(b) (6) Let

$$B = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 3 & 1 & -3 \end{pmatrix}$$

Calculate the eigenvalues and eigenvectors of B and C.

(c) (4) Let

$$C = \begin{pmatrix} 3 & 1 & 1 & 4 \\ \rho & 4 & 10 & 1 \\ 1 & 7 & 17 & 3 \\ 2 & 2 & 4 & 3 \end{pmatrix}$$

Using elementary transformations (elementary row and column operations), find the value of ρ that minimizes the rank of C.