C&O 463/663 Convex Optimization and Analysis (Fall 2007)
Assignment 3

Due 1PM (before class), on Thursday, Nov. 8, 2007

1 Cones and Theorems of the Alternative

1. Suppose that S is a ccc, i.e. a closed convex cone. S is called *pointed* if $S \cap (-S) = \{0\}$. Show that:
   $\operatorname{int}(S^+) \neq \emptyset$ iff S is pointed.

2. Let T be a polyhedral cone in $\mathbb{R}^m$, and S_i be polyhedral cones in $\mathbb{R}^{n_i}, i = 1, 2$, where S_1 is pointed. Let A_i be an $m \times n_i$ real matrix, $i = 1, 2, 3$, with $A_1 \neq 0$. Show that exactly one of the following two systems is consistent:

$$\text{(I)} \qquad A_1x^1 + A_2x^2 + A_3x^3 \in T, \quad 0 \neq x^1 \in S_1, \quad x^2 \in S_2;$$

$$\text{(II)} \qquad y \in -T^+, \quad A_1^Ty \in \operatorname{int}(S_1^+), \quad A_2^Ty \in S_2^+, \quad A_3^Ty = 0.$$

   (Hint: First show that they cannot both have a solution. Then assume that (I) is inconsistent and apply Farkas Lemma.)

3. Conclude from Item 1 that exactly one of the following two systems is consistent:

$$\text{(I)} \qquad A_1x^1 + A_2x^2 + A_3x^3 = 0, \quad 0 \neq x^1 \geq 0, \quad x^2 \geq 0;$$

$$\text{(II)} \qquad A_1^Ty > 0, \quad A_2^Ty \geq 0, \quad A_3^Ty = 0.$$

2 Normal Cones and Subdifferentials

1. (Normals to Epigraphs) For a function $f : \mathbb{E} \to (-\infty, +\infty]$ and a point $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$, calculate the normal cone $N_{\operatorname{epi} f}(\bar{x}, f(\bar{x}))$.

2. (Chain Rules) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function.

   (a) Let A be an $n \times m$ matrix. Show that the subdifferential of $F(x) := f(Ax)$ is given by
$$\partial F(x) = A^T \partial f(Ax).$$

   (b) Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth scalar function. Show that the function $F(x) := g(f(x))$ is directionally differentiable at all x, and its directional derivative is given by
$$F'(x; d) = \nabla g(f(x)) f'(x; d), \qquad \forall x, d.$$
   Furthermore, if g is convex and monotonically nondecreasing, then F is convex and its subdifferential is given by
$$\partial F(x) = \nabla g(f(x)) \partial f(x), \qquad \forall x.$$

(c) (BONUS QUESTION/OPTIONAL:) Let $f : \mathbb{E} \to \mathbb{R}$ be proper and convex and let $\mathbb{A} : \mathbb{Y} \to \mathbb{E}$ be a linear transformation. Show that

$$\partial(f \circ \mathbb{A})(x) \supset \mathbb{A}^{\mathrm{adj}}\,\partial(\mathbb{A}x), \quad \forall \mathbb{A}x \in \mathrm{dom}\,(f),$$

and equality holds if $\mathrm{int}\,(\mathrm{dom}\, f) \cap \mathbb{A}(\mathbb{Y}) \neq \emptyset$.