C&O 463/663 Convex Optimization and Analysis (Fall 2007)
Assignment 1

Due 1PM (before class), on Thursday, Oct. 4, 2007

1 Convex Sets

1. Let $\mathbb{E}$ be a Euclidean space and $S \subset \mathbb{E}$.
   (a) Show that S is convex if and only if its intersection with any line is convex.
   (b) Show that S is affine if and only if its intersection with any line is affine.
2. *(Solution set of a quadratic inequality.)* Let $C \subset \mathbb{R}^n$ be the solution set of a quadratic inequality,
   $$C = \{x \in \mathbb{R}^n : x^T A x + b^T x + c \le 0\},$$
   where $A \in \mathcal{S}^n$, the space of $n \times n$ real, symmetric matrices, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$.
   (a) Show that C is convex if $A \succeq 0$, i.e. is positive semidefinite.
   (b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbb{R}$.
   (c) *(Hyperbolic sets.* Show that the *hyperbolic* set $\{x \in \mathbb{R}^2_+ : x_1 x_2 \ge 1\}$ is convex. As a generalization, show that $\{x \in \mathbb{R}^n_+ : \Pi_{i=1}^n x_i \ge 1\}$ is convex. *Hint:* If $a, b \ge 0$ and $0 \le \theta \le 1$, then $a^\theta b^{1-\theta} \le \theta a + (1-\theta) b$.
3. *(Converse supporting hyperplane theorem)* Suppose that the set $C \subset \mathbb{E}$ is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary. Show that C is convex.
4. *(Polar Cones)* The *nonnegative polar* of a set $K \subset \mathbb{E}$ is $K^+ = \{\phi : \langle \phi, k \rangle \ge 0, \forall k \in K\}$. Show that the set K is a closed convex cone if and only if $K^{++} = K$. (Here, the second polar $K^{++} = (K^+)^+$.)

2 Convex Functions

1. *(Epigraphs.)* When is the epigraph of a function a:
   (a) halfspace?
   (b) convex cone?
   (c) polyhedron?

2. *(Convex hull or envelope of a function)* The convex hull or convex envelope of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$g(x) = \inf\{t : (x, t) \in \mathrm{conv\,epi}\, f\}.$$

Geometrically, the epigraph of g is the convex hull of the epigraph of f. Show that g is the largest convex understimator of f. In other words, show that if h is convex and satisfies $h(x) \leq f(x)$ for all x, then $h(x) \leq g(x)$ for all x.

3 Conjugate Functions

1. Derive the conjugates of the following functions.

   (a) *(Max function.)* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbb{R}^n$.

   (b) *(Sum of largest elements.)* $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$, where $x_{[i]}$ refers to the vector x sorted in nondecreasing order.

   (c) *(Largest eigenvalue.)* $f(X) = \max_{i=1}^{n} \lambda_i(X)$, where $\lambda_i(X)$ denotes the i-th largest eigenvalue of $X \in \mathcal{S}^n$.

4 Geometric Programming

1. Express the following problems as convex optimization problems.

   (a)

   $$\min_x \{\max\{p(x), q(x)\}\},$$

   where p, q are posynomials.

   (b)

   $$\min_x \exp\left(p(x)\right) + \exp\left(q(x)\right),$$

   where p, q are posynomials.

   (c)

   $$\min_{r(x) > q(x)} \frac{p(x)}{r(x) - q(x)},$$

   where p, q are posynomials, and r is a monomial.