Math 235 Assignment 11 General Review Exercises

1. Example 6 on page 410-411 shows that for the quadratic form $g(x, y) = 2x^2 + 2xy + 2y^2$, the equation $g(x, y) = 1$ gives an ellipse. Let a be a real parameter. Classify the shape of the equation $ax^2 + 2xy + y^2 = 1$ according to the range of the parameter a. Plot the curve at $a = -2$ in particular, indicating where the principal axes are.

2. From the Text, 7.1 page 446, # 7, 9 (you can check the correctness of your answers against those given on page 553-554.)

3. Use diagonalization to evaluate the indicated powers:

$$\text{(a)} \quad A^n \quad \text{for} \quad A = \begin{pmatrix} -3 & 2 \\ -12 & 7 \end{pmatrix}$$

$$\text{(b)} \quad B^{15} \quad \text{for} \quad B = \begin{pmatrix} 4 & -6 \\ 3 & -5 \end{pmatrix}$$

4. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}.$$

Show that A is diagonalizable iff either $(a - d)^2 + 4bc \neq 0$ or $a = d$ and $b = c = 0$.

5. Let $A, B \in M_{n \times n}(\mathbb{F})$ and suppose that both A and B are invertible. Prove that AB and BA have the same eigenvalues.

6. Solve the following differential equations.

$$\text{(a)} \quad \dot{\mathbf{x}} = A\mathbf{x} \qquad A = \begin{pmatrix} 12 & -51 \\ 2 & -11 \end{pmatrix}$$

$$\text{(b)} \quad \dot{\mathbf{x}} = B\mathbf{x} \qquad B = \begin{pmatrix} 3 & -11 & 16 \\ 2 & -8 & 8 \\ 1 & -3 & 2 \end{pmatrix}$$

$$\text{(c)} \quad \dot{\mathbf{x}} = C\mathbf{x} \qquad C = \begin{pmatrix} 1 & 2 \\ -5 & 3 \end{pmatrix}$$

$$\text{(d)} \quad \dot{\mathbf{x}} = D\mathbf{x} \qquad D = \begin{pmatrix} 1 & 5 \\ 5 & 3 \end{pmatrix}$$

$$\text{(e)} \quad \dot{\mathbf{x}} = E\mathbf{x} \qquad E = \begin{pmatrix} 0 & 2 & -3 \\ 0 & 4 & -5 \\ -1 & 3 & -3 \end{pmatrix}.$$

7. Find the matrix associated with each of the following quadratic forms and thus determine their nature (positive definite, negative definite, or otherwise).

   (a) $3x^2 + 5y^2 + 4xy$
   (b) $-3x^2 + 5y^2 + 4xy$
   (c) $13x^2 - 8xy + 2xz + 10y^2 - 8yz + 13z^2$
   (d) $x^2 + 4xy + 6xz + 2y^2 + 4yz + z^2$

8. Sketch the following conics in the x–y plane.

   (a) $4x^2 + 9y^2 + 12xy = 4$
   (b) $11x^2 + 6xy + 19y^2 = 80$
   (c) $2x^2 - 72xy + 23y^2 = -50$

9. Find a linear transformation $F : \mathbb{R}^3 \to \mathbb{R}^4$ such that range $F =$ $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix} \right\}$.

10. Let $T_1 : V \to V$ and $T_2 : V \to V$ be linear operators onto vector space V. Show that if $T_1 \circ T_2 = T_2 \circ T_1$, then $\ker T_1$ and $\ker T_2$ are subspaces of $\ker(T_1 \cdot T_2)$.

11. Find all eigenvalues and a basis of each eigenspace of the operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y \\ y - z \\ 2y + 4z \end{bmatrix}.$$

12. Let A and B be $n \times n$ matrices. Show that AB and BA have the same eigenvalues.

13. **Normal Matrices**

    (a) Suppose that $A \in \mathcal{M}_n(\mathcal{C})$ is normal and A has real eigenvalues. Show that A is Hermitian.

    (b) Suppose that $B \in \mathcal{M}_n(\Re)$ is *skew-symmetric*, i.e. $B^T = -B$. Show that B is normal.

    (c) Suppose that $Q \in \mathcal{M}_n(\Re)$ is *orthogonal*, i.e. $Q^T = Q^{-1}$. Show that Q is normal.

    (d) Suppose that $A \in \mathcal{M}_n(\Re)$. Show that A and A^T always share the same eigenvalues. Suppose that, in addition, A is normal. Show that A and A^T share the same eigenvectors.
    Does this mean that both A and A^T are diagonalized with the same D, P? But then why are they *not* equal in general?

14. Pending. (More may follow, if and when instructors find the time.)