ELSEVIER

Discrete Applied Mathematics 119 (2002) 79–106

DISCRETE APPLIED MATHEMATICS

Strengthened semidefinite relaxations via a second lifting for the Max-Cut problem

Miguel F. Anjos [1], Henry Wolkowicz *,[2]

Department of Combinatorics & Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ont Canada N2L 3G1

Received 29 October 1999; received in revised form 8 February 2001; accepted 19 February 2001

Abstract

In this paper we study two strengthened semidefinite programming relaxations for the Max-Cut problem. Our results hold for every instance of Max-Cut; in particular, we make no assumptions about the edge weights. We prove that the first relaxation provides a strengthening of the Goemans–Williamson relaxation. The second relaxation is a further tightening of the first one and we prove that its feasible set corresponds to a convex set that is larger than the cut polytope but nonetheless is strictly contained in the intersection of the elliptope and the metric polytope. Both relaxations are obtained using Lagrangian relaxation. Hence, our results also exemplify the strength and flexibility of Lagrangian relaxation for obtaining a variety of SDP relaxations with different properties.

We also address some practical issues in the solution of these SDP relaxations. Because Slater's constraint qualification fails for both of them, we project their feasible sets onto a lower dimensional space in a way that does not affect the sparsity of these relaxations but guarantees Slater's condition. Some preliminary numerical results are included. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Max-Cut problem; Semidefinite programming relaxations; Lagrangian relaxation; Cut polytope; Metric polytope

1. Introduction

The Max-Cut problem (MC) is a combinatorial optimization problem on undirected graphs with weights on the edges. Given such a graph, the problem consists in finding a partition of the set of vertices into two parts that maximizes the sum of the weights on the edges that have one end in each part of the partition. We consider the general

* Corresponding author. Tel.: +1-519-888-4567x5589.

E-mail addresses: manjos@stanfordalumni.org (M.F. Anjos), hwolkowi@orion.math.uwaterloo.ca (H. Wolkowicz).

[1] Research supported by an FCAR Ph.D. Research Scholarship.

[2] Research supported by the Natural Sciences and Engineering Research Council of Canada.

PII: S0166-218X(01)00266-9

case where the graph is complete and we require no restriction on the type of edge weights. So, in particular, negative or zero edge weights are permitted.

The MC problem has applications in circuit layout design and statistical physics, see e.g. [11,43]. Moreover, as a result of the celebrated work of Goemans and Williamson [23], this problem has become the flagship problem when studying applications of semidefinite programming to combinatorial optimization [20–22,49,33,27,32,12,15,30], etc. Furthermore, the MC problem is closely related to the so-called cut polytope, an important structure in the area of integer programming. The book of Deza and Laurent [17] presents many theoretical results about the cut polytope, and some connections to general integer programming are elaborated in [31].

It is well-known that MC is an NP-complete problem [37] and that it remains NP-complete for some restricted versions, see e.g. [19]. Nonetheless, some special cases can be solved efficiently. If the graph is not contractible to K_5, the complete graph on five vertices, then Barahona [9] proved that the polyhedral relaxation obtained from the triangle inequalities yields exactly the optimal value of MC. (The triangle inequalities model the constraints that for any three mutually connected vertices in the graph, it is only possible to cut either zero or two of the edges joining them.)

In this paper, we focus on the problem of obtaining tight (upper) bounds on the optimal value of MC using semidefinite programming (SDP) relaxations. Polyhedral and semidefinite relaxations for the MC problem can be obtained in different ways. All the procedures we mention are based on some form of lifting and projecting of variables between spaces of varying dimensions. Lift-and-project procedures that find polyhedral relaxations for $\{0,1\}$ programs have been studied by several authors, e.g. Balas et al. [6–8], Lovász and Schrijver [44] and Sherali and Adams [50,51].

To find semidefinite relaxations, Lovász and Schrijver [44] define a procedure, denoted N_+, that can be iterated to obtain tighter and tighter semidefinite relaxations of the convex hull of feasible integer points for $\{0,1\}$ programs. A key result is that iterating the N_+ procedure n times, where n is the number of integer variables in the problem, yields exactly the convex hull of all the integer points. For Max-Cut, n equals the number of vertices in the graph and this convex hull is usually referred to as the cut polytope. Furthermore, optimizing the appropriate linear objective function over this polytope yields exactly the optimal value of MC.

Another way to obtain semidefinite programming relaxations is via the application of the theory of moments and its dual theory, the representation of strictly positive polynomials over compact sets. The recent work of Lasserre [38] introduces a family of semidefinite programming relaxations corresponding to liftings of polynomial boolean problems into higher and higher dimensions. Lasserre presents necessary and sufficient conditions under which the optimal value of MC is attained after a finite number of such liftings.

Yet another way to derive these relaxations is through Lagrangian duality, see e.g. [52,48]. In this approach one takes the formulation MC0 (defined below) of Max-Cut and forms its Lagrangian dual. The dual of the dual yields an SDP relaxation, denoted by SDP1 in this paper. It is equivalent to the Shor relaxation [52] and the S-procedure of Yakubovitch [54,55]. One advantage of this approach over the two previously mentioned is that it is possible to choose the (possibly redundant) constraints that are included in the primal problem formulation. This choice determines the structure and properties of the resulting SDP relaxation.

Goemans and Williamson [22–24] have used the SDP1 relaxation in their algorithm for finding a cut whose weight is guaranteed to be within 14% of the weight of the maximum cut. Recently, Feige and Schechtman [18] analyzed the SDP1 relaxation with all the triangle inequalities included and constructed examples for which it is only within 11% of the weight of the maximum cut. Further approximation results for problems with general quadratic objective functions are presented in [46,56,47]. In particular, Nesterov [46,47] uses the SDP1 bound to provide estimates of μ^* for arbitrary edge weights with constant relative accuracy.

The feasible set of the SDP1 relaxation is the set of correlation matrices, or elliptope, and it has been well-studied in the literature, see e.g. [40,41] as well as the book of Deza and Laurent [17] and the references therein. Since SDP1 is a relaxation of MC, the analysis of Goemans and Williamson implies that the optimal value of SDP1 is at most 14% above the optimal value of MC. It is important to note that this result requires the assumption that there are no negative edge weights.

In this paper we study two strengthened SDP relaxations for MC. Our results hold for every instance of MC; in particular, we make no assumptions about the edge weights. The basic SDP1 relaxation may be obtained by applying Lagrangian relaxation and thereby lifting the quadratic boolean formulation of MC from the space $\Re^n$ into $\mathcal{S}^n$, the space of $n \times n$ symmetric matrices. These new relaxations are obtained by applying a second lifting, i.e. by suitably formulating MC in the space $\mathcal{S}^n$ and then lifting the problem into $\mathcal{S}^{[n(n+1)/2]+1}$.

We prove that the first relaxation, denoted SDP2, is a strengthening of the Goemans–Williamson relaxation. The second relaxation, denoted SDP3, [3] is a further tightening of SDP2. We prove that its feasible set corresponds, via a suitable projection back to $\mathcal{S}^n$, to a convex set F_n that is larger than the cut polytope but is strictly contained in the intersection of the elliptope and the metric polytope. This illustrates the strength of Lagrangian relaxation for obtaining different SDP relaxations with different properties.

We also address some practical issues in the solution of these SDP relaxations. Because Slater's constraint qualification fails for both SDP2 and SDP3, we can project their feasible sets onto a lower dimensional space. This is done in a simple way that does not affect the sparsity of these relaxations and ensures that Slater's condition is satisfied after the projection. We include some numerical results.

This paper is structured as follows. In the remainder of this section we review the definitions and notation to be used in this paper and we sketch the application of Lagrangian relaxation to derive the basic relaxation SDP1. In Section 2, we derive the SDP2 strengthened relaxation and prove several of its properties. In Section 3, we

[3] A relaxation equivalent to SDP3 was presented by Michel Goemans at the fourth International Conference on High Performance Optimization Techniques, June 1999, Rotterdam, Netherlands.

derive the tighter relaxation SDP3 and prove that the projection of its feasible set into $\mathcal{S}^n$ gives a strict subset of the intersection of the elliptope and the metric polytope. Since neither of these two relaxations has a strictly feasible point, in Section 4, we study the geometry of their feasible sets and find a projection onto the minimal face of the positive semidefinite cone where we verify that Slater's constraint qualification holds. Finally, in Section 5, we report some numerical results obtained by solving the projected relaxations.

1.1. Max-Cut formulations and relaxations

Following [45], we can formulate the MC problem as follows. Let the given graph G have vertex set $\{1,\dots,n\}$ and let it be described by its weighted adjacency matrix $A(G)$. We tacitly assume that the graph in question is complete (if not, missing edges can be given weight 0 to complete the graph) and that the edge set is not empty, so $A(G)\neq 0$. Let L denote the *Laplacian matrix* associated with the graph; hence $L:=\mathrm{Diag}(A(G)\cdot e) - A(G)$, where the linear operator Diag returns a diagonal matrix with diagonal formed from the vector given as its argument, and e denotes the vector of all ones. Let the vector $v\in\{\pm 1\}^n$ represent any cut in the graph via the interpretation that the sets $\{i\colon v_i=+1\}$ and $\{i\colon v_i=-1\}$ form a partition of the vertex set of the graph. Then we can formulate MC as

$$(\mathrm{MC0})\quad \mu^* = \max \quad \tfrac{1}{4}v^{\mathrm{T}}Lv \qquad \text{s.t.} \quad v_i^2=1,\quad i=1,\dots,n,$$

where μ^* denotes the optimal value of MC. Define $Q:=\frac{1}{4}L$ and consider the change of variable $X:=vv^{\mathrm{T}}, v\in\{\pm 1\}^n$. Then $v^{\mathrm{T}}Qv=\mathrm{trace}\, QX$, and an equivalent formulation for MC is

$$\begin{array}{llll}(\mathrm{MC1}) & \mu^* = \max & \mathrm{trace}\, QX & \\ & \text{s.t.} & \mathrm{diag}(X)=e, & \\ & & \mathrm{rank}(X)=1, & \\ & & X\succcurlyeq 0, \quad X\in\mathcal{S}^n, & \end{array}$$

where $\mathrm{diag}(\cdot)$ denotes the linear operator that returns a vector with the diagonal elements of its matrix argument, $X\succcurlyeq 0$ denotes that X is positive semidefinite and $\mathcal{S}^n$ denotes the space of $n\times n$ symmetric matrices. This space has dimension $t(n):=n(n+1)/2$ and is endowed with the trace inner product $\langle A,B\rangle:=\mathrm{trace}\, AB$.

Denote the feasible set of MC1, less the rank constraint, by

$$\mathcal{E}_n:=\{X\in\mathcal{S}^n\colon\ \mathrm{diag}(X)=e, X\succcurlyeq 0\}.$$

The set $\mathcal{E}_n$ is the elliptope studied in [40,41] and it is a convex relaxation of the feasible set of problem MC1 since we dropped the rank constraint. With this notation, let us define the semidefinite programming problem SDP1:

$$(\mathrm{SDP1})\quad \nu_1^* = \max \quad \mathrm{trace}\, QX \qquad \text{s.t.} \quad X\in\mathcal{E}_n.$$

This SDP relaxation is well-known and has been studied in e.g. [16,23,42]. Goemans and Williamson [23] provided estimates for the quality of the SDP1 bound for MC. They proved that the optimal value of this relaxation is at most 14% above the value of the maximum cut, provided there are no negative edge weights. More precisely,

$$\mu^* \geqslant \alpha \nu_1^*,$$

where $\alpha := \min_{0 \leqslant \theta \leqslant \pi} 2/\pi\theta/(1 - \cos\theta) \approx 0.87856$, and since $1/\alpha \leqslant 1.14$, we have that $\nu_1^* \leqslant 1.14\mu^*$. Furthermore, by randomly rounding a solution to the SDP relaxation, they obtain a 0.878-approximation algorithm, i.e. an algorithm that produces a cut with value at least 0.878 times the optimal value. (Note that Håstad [26] proved that it is NP-hard to find a ρ-approximation algorithm for Max-Cut with ρ greater than 0.9412.)

Other convex relaxations of the feasible set of MC1 have been proposed in the literature. The smallest convex set containing all the matrices X which are feasible for MC1 is their convex hull, called the *cut polytope*:

$$C_n := \mathrm{Conv}\{X : X = vv^{\mathrm{T}}, v \in \{\pm 1\}^n\}.$$

Optimizing trace QX over C_n would yield exactly μ^*, but an efficient description of the cut polytope is not known. A well-known relaxation of the cut polytope is the *metric polytope* M_n, defined as the set of all matrices satisfying the triangle inequalities:

$$M_n := \{X \in \mathcal{S}^n:\ \mathrm{diag}(X) = e,\ \text{and}\ X_{ij} + X_{ik} + X_{jk} \geqslant -1, X_{ij} - X_{ik} - X_{jk} \geqslant -1,$$
$$-X_{ij} + X_{ik} - X_{jk} \geqslant -1, -X_{ij} - X_{ik} + X_{jk} \geqslant -1,\ \forall\, 1 \leqslant i < j < k \leqslant n\}.$$

The triangle inequalities model the following constraints: for any three mutually connected vertices in the graph, it is only possible to cut either zero or two of the edges joining them. In fact, the triangle inequalities are sufficient to describe the cut polytope for graphs with less than five vertices, i.e. $C_n = M_n$ for $n \leqslant 4$; however, $C_n \subsetneq M_n$ for $n \geqslant 5$, see for example [17]. Nonetheless, if the graph G is not contractible to K_5, the complete graph on 5 vertices, then Barahona [10] proved that the linear programming problem

$$\begin{array}{ll} \max & \mathrm{trace}\, QX \\ \text{s.t.} & X \in M_n \end{array}$$

has optimal value equal to μ^*.

1.2. SDP1 via Lagrangian relaxation — first lifting

The SDP1 relaxation can be obtained by taking the Lagrangian dual of the Lagrangian dual of the formulation MC0 [52,48]. The Lagrangian dual of MC0 is

$$\mu^* \leqslant \nu_1^* := \min_y \max_v v^{\mathrm{T}}Qv - v^{\mathrm{T}}(\mathrm{Diag}\, y)v + e^{\mathrm{T}}y.$$

The inner maximization has a hidden constraint, i.e. the quadratic is bounded above only if its Hessian is negative semidefinite. This is equivalent to the following SDP:

$$\begin{array}{lll} \nu_1^* = \min & e^{\mathrm{T}}y \\ \text{s.t.} & \mathrm{Diag}\, y \succcurlyeq Q. \end{array}$$

Slater's (strict feasibility) constraint qualification holds for this problem. Therefore, its Lagrangian dual has the same optimal value and is precisely SDP1:

$$\begin{array}{rll} \mu^* \leqslant v_1^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e, \\ & & X \succcurlyeq 0. \end{array}$$

Whenever the objective function or the constraints in MC0 contain a linear term, negative semidefiniteness of the Hessian is not sufficient for boundedness of the quadratic; feasibility of the stationarity condition is also needed. Alternatively, one can homogenize and use strong duality of the trust region subproblem [53]. The latter technique is used below.

2. First strengthened SDP relaxation

The derivation of SDP2 begins by adding to MC1 the *redundant* quadratic constraints

- $X^2 - nX = 0$ and
- $X \circ X = E$, where $\circ$ denotes the Hadamard (elementwise) product of matrices and E denotes the matrix of all ones.

One motivation for adding these redundant constraints to the formulation MC1 is that we will be using Lagrangian duality to obtain new SDP relaxations. To obtain tighter bounds we therefore want the duality gap incurred in the process to be as small as possible. The interest in these redundant quadratic constraints comes from the results in [4,5] where the addition of redundant constraints of this type was shown to guarantee strong duality for certain problems where duality gaps can exist.

The validity of the constraint $X^2 - nX = 0$ follows from the observation that $X^2 = vv^{\mathrm{T}}vv^{\mathrm{T}}$ and $v^{\mathrm{T}}v = n$ for all $v \in \{\pm 1\}^n$. Since we can simultaneously diagonalize X and X^2, the eigenvalues of X must satisfy $\lambda^2 - n\lambda = 0$, which implies that the only eigenvalues of X are 0 and n. This shows that the constraint $X \succcurlyeq 0$ becomes redundant and may be removed. Moreover, since the diagonal constraint implies that the trace of X is n, we conclude that X must be rank-one and the rank constraint can also be removed. Finally, if $v \in \{\pm 1\}^n$ then all the entries of $X = vv^{\mathrm{T}}$ are ± 1 and clearly $X \circ X = E$ holds. (In fact, by Theorem 2.2, this last constraint together with $X \succcurlyeq 0$ also implies that X is rank-one).

The resulting problem MC2 is thus another formulation of MC:

$$\begin{array}{rll} (\text{MC2})\ \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X) = e, \\ & & X \circ X = E, \\ & & X^2 - nX = 0. \end{array}$$

We now present two different derivations of the strengthened relaxation SDP2. The first derivation follows the procedure in [48] which was illustrated in Section 1.2, while the second derivation is a direct second lifting using the motivation that cuts correspond to rank-one matrices in the strengthened relaxation. Although the second derivation is simpler and can be done independently, we also include the first one because it gives insight on how the choice of (possibly redundant) constraints determines the SDP relaxation we obtain from the Lagrangian dual. Furthermore, once the result of the first derivation is obtained, the second derivation of the same SDP becomes obvious. But it is not clear how to directly derive an SDP that has not yet been formulated. (Note that the equivalence of the two derivations follows from Theorem 9 of [48] and the discussion preceding the Theorem therein.)

2.1. Second lifting via Lagrangian duality

In this section we follow the "recipe" presented in [48]. This recipe can be summarized as:

1. Add as many redundant quadratic constraints as possible;
2. Take the Lagrangian dual of the Lagrangian dual;
3. Remove redundant constraints and project the feasible set of the resulting SDP to guarantee strict feasibility.

This section presents only the first two steps of this recipe plus the removal of redundant constraints. The projection is applied in Section 4 where we study the geometry of the feasible sets of our relaxations.

To efficiently apply Lagrangian relaxation and not lose the information from the linear constraint, we need to replace the constraint with the norm constraint $||\mathrm{diag}(X)-e||^2=0$ and homogenize the problem. We then lift this matrix problem into a higher dimensional matrix space.

To keep the dimension as low as possible, we take advantage of the symmetry of X. Recall that $t(i)=i(i+1)/2$ and define sMat to be the linear operator that, given a vector $x\in\mathfrak{R}^{t(n)}$, returns a matrix $X\in\mathscr{S}^n$ obtained by filling in columnwise the upper triangular part of X with the $t(n)$ components of x and completing the strictly lower triangle by symmetry. Thus, we rewrite MC2 as

$$
\begin{array}{llll}
(\mathrm{MC2}) & \mu^* = & \max & \mathrm{trace}(Q\,\mathrm{sMat}(x))y_0 \\
& & \text{s.t.} & \mathrm{diag}(\mathrm{sMat}(x))^{\mathrm{T}}\mathrm{diag}(\mathrm{sMat}(x)) - 2e^{\mathrm{T}}\mathrm{diag}(\mathrm{sMat}(x))y_0 + n = 0, \\
& & & E - \mathrm{sMat}(x)\circ\mathrm{sMat}(x) = 0, \\
& & & \mathrm{sMat}(x)^2 - n\,\mathrm{sMat}(x)y_0 = 0, \\
& & & 1 - y_0^2 = 0, \\
& & & x\in\mathfrak{R}^{t(n)}, \quad y_0\in\mathfrak{R}.
\end{array}
$$

Note that this problem is equivalent to the previous formulation since we can change X to $-X$ if $y_0=-1$ is optimal.

We now take the Lagrangian dual of MC2. Introducing Lagrange multipliers $w,t\in\mathfrak{R}$ and $T,S\in\mathscr{S}^n$, the dual is

$$
\begin{aligned}
v_2^* := \min_{t,w,T,S}\max_{x,y_0}\{&\mathrm{trace}(Q\,\mathrm{sMat}(x))y_0 \\
&+ w(\mathrm{diag}(\mathrm{sMat}(x))^{\mathrm{T}}\,\mathrm{diag}(\mathrm{sMat}(x)) - 2e^{\mathrm{T}}\,\mathrm{diag}(\mathrm{sMat}(x))y_0 + n)
\end{aligned}
$$

$$+\operatorname{trace} T(E - \operatorname{sMat}(x)\circ\operatorname{sMat}(x))$$
$$+\operatorname{trace} S((\operatorname{sMat}(x))^2 - n\operatorname{sMat}(x)y_0) + t(1-y_0^2)\}.$$

Note that moving the constraint $y_0^2=1$ into the Lagrangian does not increase the duality gap, since the Lagrangian relaxation of the trust-region subproblem is tight [53].

Applying the steps sketched in Section 1.2 for deriving SDP1, we take the dual of this dual and obtain the SDP2 relaxation (see [2] for details). We shall make use of the following linear operators:

- $\operatorname{Diag}(v)$ forms a diagonal matrix with the vector v on the diagonal;
- svec is the inverse of sMat, i.e. it forms a $t(n)$-vector columnwise from an $n\times n$ symmetric matrix while ignoring the strictly lower triangular part of the matrix;
- dsvec acts like svec but multiplies by 2 the off-diagonal entries of its (symmetric) matrix argument;
- Mat forms an $n\times n$ matrix columnwise from an n^2-vector;
- vec is the inverse of Mat;
- $\operatorname{vsMat}(x) := \operatorname{vec}(\operatorname{sMat}(x))$.

First, the inner maximization of the Lagrangian dual of MC2 is an unconstrained pure quadratic maximization, therefore, its optimal value is infinity unless the Hessian is negative semidefinite in which case $x=0,\ y_0=0$ is optimal. Thus we need to calculate the Hessian.

Using $\operatorname{trace} Q\operatorname{sMat}(x) = x^{\mathrm{T}}\operatorname{dsvec}(Q)$, and pulling out a 2 for convenience, we get the constant part (no Lagrange multipliers) of the Hessian:

$$2H_c := 2\begin{pmatrix} 0 & \frac{1}{2}\operatorname{dsvec}(Q)^{\mathrm{T}} \\ \frac{1}{2}\operatorname{dsvec}(Q) & 0 \end{pmatrix}.$$

The nonconstant part of the Hessian is made up of a linear combination of matrices, i.e. it is a linear operator on the Lagrange multipliers. Again for notational convenience, we let $\mathscr{H}(w,T,S,t)$ denote the *negative* of the nonconstant part of the Hessian, and we split it into four linear operators with the factor 2:

$$\begin{aligned}
2\mathscr{H}(w,T,S,t) &:= 2\mathscr{H}_1(w) + 2\mathscr{H}_2(T) + 2\mathscr{H}_3(S) + 2\mathscr{H}_4(t) \\
&:= 2w\begin{pmatrix} 0 & (\operatorname{dsvec}\operatorname{Diag} e)^{\mathrm{T}} \\ (\operatorname{dsvec}\operatorname{Diag} e) & -\operatorname{sdiag}^*\operatorname{sdiag} \end{pmatrix} \\
&\quad + 2\begin{pmatrix} 0 & 0 \\ 0 & \operatorname{dsvec}(T\circ\operatorname{sMat}) \end{pmatrix} \\
&\quad + 2\begin{pmatrix} 0 & \frac{n}{2}\operatorname{dsvec}(S)^{\mathrm{T}} \\ \frac{n}{2}\operatorname{dsvec}(S) & (\operatorname{Mat}\operatorname{vsMat})^* S(\operatorname{Mat}\operatorname{vsMat}) \end{pmatrix} \\
&\quad + 2t\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Thus, after cancelling the factor of 2 on both sides of the constraint, we get the semidefinite program

$$\begin{aligned} v_2^* = \min \quad & nw + \operatorname{trace} ET + \operatorname{trace} 0S + t \\ \text{s.t.} \quad & \mathscr{H}(w,T,S,t) \succeq H_c. \end{aligned}$$

If we take T sufficiently positive definite and t sufficiently large, then we can guarantee Slater's constraint qualification. Therefore, the dual of this SDP has the same optimal value v_2^* and it provides the strengthened SDP relaxation of MC:

$$\text{(SDP2)} \quad \begin{aligned} v_2^* = \max \quad & \operatorname{trace} H_Q Y \\ \text{s.t.} \quad & \mathscr{H}_1^*(Y) = n, \\ & \mathscr{H}_2^*(Y) = E, \\ & \mathscr{H}_3^*(Y) = 0, \\ & \mathscr{H}_4^*(Y) = 1, \\ & Y \succeq 0, \quad Y \in \mathscr{S}^{t(n)+1}, \end{aligned}$$

where

$$H_Q := \begin{pmatrix} 0 & \frac{1}{2}\operatorname{dsvec}(Q)^{\mathrm{T}} \\ \frac{1}{2}\operatorname{dsvec}(Q) & 0 \end{pmatrix}.$$

To express the linear operators $\mathscr{H}_i^*(Y)$, $i = 1,2,3,4$, let us index the rows of Y by $0,1,\ldots,t(n)$ and partition it as

$$Y = \begin{pmatrix} Y_{00} & x^{\mathrm{T}} \\ x & \bar{Y} \end{pmatrix},$$

where $\bar{Y} \in \mathscr{S}^{t(n)}$. Then

$$\mathscr{H}_1^*(Y) = 2\operatorname{svec}(I_n)^{\mathrm{T}} x - \operatorname{trace} \operatorname{Diag}(\operatorname{svec}(I_n))\bar{Y},$$

$$\mathscr{H}_2^*(Y) = \operatorname{sMat} \operatorname{diag}(\bar{Y}),$$

$$\mathscr{H}_3^*(Y) = n \operatorname{sMat}(x) - (\operatorname{Mat} \operatorname{vsMat})\bar{Y}(\operatorname{Mat} \operatorname{vsMat})^*,$$

$$\mathscr{H}_4^*(Y) = Y_{00}.$$

The constraints $\mathscr{H}_2^*(Y) = E$ and $\mathscr{H}_4^*(Y) = 1$ are equivalent to $\operatorname{diag}(Y) = e$. Also, $\mathscr{H}_1^*(Y)$ is twice the sum of the elements in the first row of Y corresponding to the positions of the diagonal of $\operatorname{sMat}(x)$ minus the sum of the same elements in the diagonal of $\bar{Y}$. The constraint $\mathscr{H}_1^*(Y) = n$ implies that $Y_{0,t(i)} = 1 \ \forall i = 1,\ldots,n$, and thus $\operatorname{diag}(\operatorname{sMat}(x)) = e$ holds.

The constraint $\mathscr{H}_3^*(Y) = 0$ implies immediately that if Y is feasible for SDP2, then $\operatorname{sMat}(x)$ is positive semidefinite (and in fact feasible for SDP1). This is proved in Lemma 2.1.

2.2. Direct second lifting

We can derive the SDP2 relaxation directly from MC2 using the rank-one relationship

$$Y \cong \begin{pmatrix} y_0 \\ x \end{pmatrix} (y_0 \quad x^{\mathrm{T}}), \quad X = \mathrm{sMat}(x).$$

Using this approach we express the constraints that the elements of X are ± 1 and $\mathrm{diag}(X) = e$ as

$$\mathrm{diag}(Y) = e \quad \text{and} \quad Y_{0,t(i)} = 1, \; i = 1, \ldots, n.$$

We also express the $t(n+1)$ constraints from $X^2 - nX = 0$. The constraints corresponding to equating the diagonal entries become redundant (see [2] for details). After they are removed, the result is the SDP relaxation

$$\begin{array}{llll} (\mathrm{SDP2}) & v_2^* = & \max & \mathrm{trace}\, H_Q Y \\ & & \text{s.t.} & \mathrm{diag}(Y) = e, \\ & & & Y_{0,t(i)} = 1, \quad i = 1, \ldots, n, \\ & & & Y_{0,T(i,j)} = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} Y_{T(i,k),T(k,j)} \quad \forall 1 \leqslant i < j \leqslant n, \\ & & & Y \succcurlyeq 0, \; Y \in \mathcal{S}^{t(n)+1}, \end{array}$$

where

$$T(i,j) := \begin{cases} t(j-1) + i & \text{if } i \leqslant j, \\ t(i-1) + j & \text{otherwise.} \end{cases}$$

(Recall that $t(i) = i(i+1)/2$, so $T(i,i) = t(i)$.)

The first two sets of constraints imply that the 2×2 leading principal minor of any Y feasible for SDP2 is all ones. Hence, every feasible Y for SDP2 is singular. In Section 4, we shall exploit this fact to project the feasible set of SDP2 onto a lower dimensional face of the positive semidefinite cone and thus reduce the number of variables in the SDP relaxation.

2.3. Properties of the first strengthened relaxation

One surprising result is that the matrix obtained by applying sMat to the first row of a feasible Y is positive semidefinite, even though this nonlinear constraint was discarded in the construction of MC2 and there could be a duality gap between SDP2 and MC2.

Lemma 2.1. *Suppose that Y is feasible for SDP2. Then*

$$\mathrm{sMat}(Y_{1:t(n),0}) \succcurlyeq 0$$

and so is feasible for SDP1.

Proof. Using the partition

$$Y = \begin{pmatrix} 1 & x^{\mathrm{T}} \\ x & \bar{Y} \end{pmatrix},$$

we see that $\bar{Y}$ is positive semidefinite. Rewriting the constraint $\mathscr{H}_3^*(Y) = 0$ as

$$\mathrm{sMat}(x) = \frac{1}{n}(\mathrm{Mat}\,\mathrm{vsMat})\bar{Y}(\mathrm{Mat}\,\mathrm{vsMat})^*,$$

we see $\mathrm{sMat}(x)$ is a congruence of $\bar{Y}$. The result follows. □

Consequently, the relaxation SDP2 is a strengthening of SDP1.

Theorem 2.1. *The optimal values of SDP1 and SDP2 satisfy*

$$v_2^* \leqslant v_1^*.$$

Proof. Suppose that

$$Y^* = \begin{pmatrix} 1 & x^{*\mathrm{T}} \\ x^* & \bar{Y}^* \end{pmatrix}$$

solves SDP2. From Lemma 2.1, $X^* := \mathrm{sMat}(x^*)$ is feasible for SDP1, therefore

$$\begin{aligned} v_2^* &= \mathrm{trace}\, H_Q Y^* \\ &= (\mathrm{dsvec}\, Q)^{\mathrm{T}} x^* \\ &= \mathrm{trace}\, QX^* \\ &\leqslant v_1^*. \quad \square \end{aligned}$$

We can also look at the added constraint $X \circ X = E$. Even though it does not imply $X \succcurlyeq 0$, it is interesting to note that adding only this constraint to SDP1 yields a problem equivalent to MC. This follows from the following theorem that gives a characterization for all the $\{\pm 1\}$-matrices in the positive semidefinite cone: they are exactly the rank-one matrices formed by the outer product of some $\{\pm 1\}$ n-vector with itself. (This theorem follows as a Corollary to [34, Theorem 5.3.4]. [4] We include a simple independent proof.)

Theorem 2.2. *Let X be an $n \times n$ symmetric matrix. Then*

$$X \succcurlyeq 0, \;\; X \in \{\pm 1\}^{n \times n} \quad \textit{if and only if} \quad X = xx^{\mathrm{T}}, \quad \textit{for some } x \in \{\pm 1\}^n.$$

Proof. Showing sufficiency is straightforward: if $X = xx^{\mathrm{T}}$ then for any $y \in \Re^n$, we have

$$y^{\mathrm{T}} X y = ||x^{\mathrm{T}} y||^2 \geqslant 0,$$

hence X is positive semidefinite.

[4] Thanks to Yin Zhang for this reference.

To prove necessity, first observe that if X is symmetric, $X \in \{\pm 1\}^{n\times n}$, and $X \succcurlyeq 0$, then all the diagonal entries of X equal 1.

If $n=2$, the possibilities for X are

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and it is easily checked that both are positive semidefinite and rank-one.

For $n \geqslant 3$, we argue by contradiction. Suppose $X \in \{\pm 1\}^{n\times n}$ and $X \succcurlyeq 0$ but X is not rank-one. Let $X=(x_{ij})$ and let x_j denote the jth column of X. Then without loss of generality (permuting columns if necessary) the first two columns of X are linearly independent, therefore, $x_1 \neq x_2$ and $x_1 \neq -x_2$. Hence, again without loss of generality (permuting rows if necessary), $x_{11}=x_{12}=1$, and $x_{31}=-x_{32}=-1$ or $x_{31}=-x_{32}=1$. Thus, the top left 3×3 principal submatrix of X is either

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Since the determinants of these matrices are negative, we have a contradiction to $X \succcurlyeq 0$. Hence, X is rank-one. Since $\operatorname{diag}(X)=e$, the result follows. □

3. Second strengthened SDP relaxation

The second strengthened relaxation is obtained by adding more redundant quadratic constraints to MC2 and obtaining a new formulation MC3. The dual of MC3 will be an even tighter relaxation since an increase in the number of Lagrange multipliers gives us a better bound.

Recall the change of variable $X=vv^{\mathrm{T}}$. Since $X_{ij}=v_iv_j$ and $v_k^2=1$ for $k=1,\dots,n$, the constraints

$$X_{ij}=v_iv_j=v_iv_k^2v_j=v_iv_k\cdot v_kv_j=X_{ik}\cdot X_{kj}$$

also hold for every rank-one X corresponding to a cut. In fact, there is an interesting connection between these constraints and the metric polytope. This connection is used in the proof of Theorem 3.1.

Adding these constraints to MC2, we obtain the formulation MC3:

$$\begin{array}{lll} (\mathrm{MC3})\ \mu^* = & \max & \operatorname{trace} QX \\ & \text{s.t.} & \operatorname{diag}(X)=e, \\ & & X\circ X=E, \\ & & X^2-nX=0, \\ & & X_{ij}=X_{ik}X_{kj} \quad \forall 1 \leqslant i,j,\ k \leqslant n. \end{array}$$

Taking the dual of the dual of MC3 (and removing redundant constraints in the resulting SDP) yields the relaxation SDP3 defined below.

Alternatively, we can motivate SDP3 by considering that the rank-one matrices $X = vv^{\mathrm{T}}$, $v \in \{\pm 1\}^n$ have all their entries equal to ± 1. Hence, the corresponding matrices Y feasible for SDP2 have all their entries in the first row and column equal to ± 1. Now consider the following constraints from SDP2:

$$Y_{0,T(i,j)} = \frac{1}{n}\sum_{k=1}^{n} Y_{T(i,k),T(k,j)} \quad \forall 1 \leqslant i < j \leqslant n$$

for

$$Y = \begin{pmatrix} 1 & x^{\mathrm{T}} \\ x & \bar{Y} \end{pmatrix} \quad \text{and} \quad x = \operatorname{svec}(vv^{\mathrm{T}}).$$

The entry $Y_{0,T(i,j)}$ is in the first row of Y and therefore it is equal to 1 in magnitude. The corresponding constraint states that it must be equal to the average of n entries in the block $\bar{Y}$. But each of these n entries has magnitude at most 1, so for equality to hold, they must all have magnitude equal to 1, and in fact they must all equal $Y_{0,T(i,j)}$.

Either approach yields the relaxation SDP3:

$$\begin{array}{llll} (\text{SDP3}) & v_3^* = & \max & \operatorname{trace} H_Q Z \\ & & \text{s.t.} & \operatorname{diag}(Z) = e, \\ & & & Z_{0,t(i)} = 1, \quad i = 1,\ldots,n, \\ & & & Z_{0,T(i,j)} = Z_{T(i,k),T(k,j)} \quad \forall k, \ \forall 1 \leqslant i < j \leqslant n, \\ & & & Z \succcurlyeq 0, \ Z \in \mathscr{S}^{t(n)+1}. \end{array}$$

3.1. Properties of the second strengthened relaxation

Let us define the projection of the feasible set of SDP3 onto $\mathscr{S}^n$ as

$$F_n := \{X \in \mathscr{S}^n \colon X = \operatorname{sMat}(Z_{1:t(n),0}),\ Z \text{ feasible for SDP3}\}.$$

Since the feasible set of SDP3 is convex and compact, and since F_n is its image under a linear transformation, it follows that F_n is also convex and compact. Also, it is straightforward to verify that F_n contains the cut polytope.

Lemma 3.1. $C_n \subseteq F_n$.

Since every Z feasible for SDP3 is feasible for SDP2, by Lemma 2.1 we have:

Corollary 3.1. $F_n \subseteq \mathscr{E}_n$.

By Lemma 3.1, we observe that $\mu^* \leqslant v_3^* \leqslant v_2^* \leqslant v_1^*$. We now prove an additional property of SDP3 that is not inherited from SDP2, namely that the matrices in F_n also satisfy all the triangle inequalities.

Theorem 3.1. $F_n \subseteq M_n$.

Proof. Suppose $X \in F_n$, then $X = \mathrm{sMat}(Z_{1:t(n),0})$ for some Z feasible for SDP3. Since $Z_{0,t(i)} = 1\ \forall i$, it follows that $\mathrm{diag}(X) = e$ holds.

Given i, j, k such that $1 \leqslant i < j < k \leqslant n$, let $Z_{i,j,k}$ denote the 4×4 principal minor of Z corresponding to the indices $0, T(i,j), T(i,k), T(j,k)$. Let $a = X_{ij} = Z_{0,T(i,j)}$, $b = X_{ik} = Z_{0,T(i,k)}$, $c = X_{jk} = Z_{0,T(j,k)}$. Then

$$Z_{i,j,k} = \begin{pmatrix} 1 & a & b & c \\ a & 1 & c & b \\ b & c & 1 & a \\ c & b & a & 1 \end{pmatrix}$$

since $\mathrm{diag}(Z) = e$ and

$$Z_{0,T(i,j)} = Z_{T(i,k),T(k,j)}, \quad Z_{0,T(i,k)} = Z_{T(i,j),T(j,k)}, \quad Z_{0,T(j,k)} = Z_{T(j,i),T(i,k)}.$$

Now

$$\begin{aligned} Z_{i,j,k} \succcurlyeq 0 &\Leftrightarrow \begin{pmatrix} 1 & c & b \\ c & 1 & a \\ b & a & 1 \end{pmatrix} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} (a \quad b \quad c) \succcurlyeq 0 \\ &\Leftrightarrow \begin{pmatrix} 1-a^2 & c-ab & b-ac \\ c-ab & 1-b^2 & a-bc \\ b-ac & a-bc & 1-c^2 \end{pmatrix} \succcurlyeq 0 \\ &\Rightarrow e^{\mathrm{T}} \begin{pmatrix} 1-a^2 & c-ab & b-ac \\ c-ab & 1-b^2 & a-bc \\ b-ac & a-bc & 1-c^2 \end{pmatrix} e \geqslant 0. \end{aligned}$$

Hence,

$$\begin{aligned} Z_{i,j,k} \succcurlyeq 0 &\Rightarrow 3 - (a+b+c)^2 + 2(a+b+c) \geqslant 0 \\ &\Leftrightarrow \gamma^2 - 2\gamma - 3 \leqslant 0, \quad \text{where } \gamma := a+b+c \\ &\Leftrightarrow (\gamma - 3)(\gamma + 1) \leqslant 0 \\ &\Leftrightarrow -1 \leqslant \gamma \leqslant 3 \\ &\Rightarrow a+b+c \geqslant -1. \end{aligned}$$

Therefore, $X_{ij} + X_{ik} + X_{jk} \geqslant -1$ holds for X.

Since multiplication of row and column i of $Z_{i,j,k}$ by -1 will not affect the positive semidefiniteness of $Z_{i,j,k}$, multiplying the two rows and two columns of $Z_{i,j,k}$ with indices $T(i,k)$ and $T(j,k)$ and applying the same argument to the resulting matrix, we obtain $X_{ij} - X_{ik} - X_{jk} \geqslant -1$. Similarly, the inequalities $-X_{ij} + X_{ik} - X_{jk} \geqslant -1$ and $-X_{ij} - X_{ik} + X_{jk} \geqslant -1$ also hold. □

We have thus proved the following:

Corollary 3.2. $C_n \subseteq F_n \subseteq \mathscr{E}_n \cap M_n$.

In Section 3.1.2, we will prove that the inclusions are in fact strict for $n \geqslant 5$. However, because we do not have an explicit description of F_n, first we need to address the issue of testing for membership in F_n. This is the focus of the next section.

3.1.1. Testing for membership in F_n

The set F_n is defined as the image of the feasible set of SDP3 under the linear mapping sMat applied to the first row of every feasible matrix in SDP3. It is not clear how to give an explicit description of F_n, but given $X \in \mathcal{S}^n$, the question of determining whether $X \in F_n$ can be expressed as:

Given $X \in \mathcal{S}^n$ satisfying $\mathrm{diag}(X) = e$, does there exist a matrix Z feasible for SDP3 such that $\mathrm{sMat}(Z_{1:t(n),0}) = X$?

In this question, only a subset of the elements of Z are specified, namely the diagonal, the first row and column and the elements fixed by the rank-two constraints. The remaining elements are considered "free" and we ask whether it is possible to choose them in such a way that the resulting matrix Z is positive semidefinite. This problem is an instance of the positive semidefinite matrix completion problem, which has been extensively studied (see e.g. [25,39,35]).

We can associate with the partial matrix Z a finite undirected graph $G_Z = (V_Z, E_Z)$ as follows: let the vertex set be $V_Z := \{0, 1, \ldots, t(n)\}$ and let the edge set E_Z contain the edge (i, j) if and only if the entry $Z_{i,j}$ is fixed. Then G_Z is said to be *chordal* if every cycle of length $\geqslant 4$ has a chord, i.e. an edge between two non-consecutive vertices. Grone et al. [25] showed that if the diagonal entries of Z are specified and the principal minors composed of fixed entries are all non-negative, then, if the graph G_Z is chordal, a positive semidefinite completion necessarily exists. In our case, however, it is easy to see that the graph G_Z is not chordal for $n \geqslant 4$. It suffices to consider the cycle of length 4 depicted in Fig. 1; since $(T(i,j), T(k,l)) \notin E_Z$ and $(T(i,k), T(j,l)) \notin E_Z$, we see that the cycle has no chords. So we must follow a different approach.

Johnson et al. [36] present an interior-point method for finding an approximate completion, if a completion exists. We use this approach to test membership in F_n. Specifically, we proceed as follows: Given $X \in \mathcal{S}^n$ with $\mathrm{diag}(X) = e$, let $x = \mathrm{svec}(X)$ and let $A \in \mathcal{S}^{t(n)+1}$ be some matrix which satisfies $\mathrm{sMat}(A_{1:t(n),0}) = X$ and furthermore satisfies all the constraints of SDP3, except (possibly) for the positive semidefiniteness

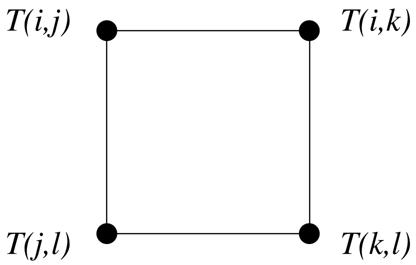

Fig. 1. A cycle of length 4 with no chord in the graph G_Z of Z.

constraint. Define $H \in \mathscr{S}^{t(n)+1}$ to be the $\{0,1\}$-matrix satisfying $H_{ij}=0$ if A_{ij} is "free", and $H_{ij}=1$ otherwise.

For example, if $X=(X_{ij})$ is 3×3, one possible choice of A is

$$A=\begin{pmatrix} 1 & 1 & X_{12} & 1 & X_{13} & X_{23} & 1\\ 1 & 1 & 0 & 0 & 0 & 0 & 0\\ X_{12} & 0 & 1 & 0 & X_{23} & X_{13} & 0\\ 1 & 0 & 0 & 1 & 0 & 0 & 0\\ X_{13} & 0 & X_{23} & 0 & 1 & X_{12} & 0\\ X_{23} & 0 & X_{13} & 0 & X_{12} & 1 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where the "free" entries are filled with zeros. The corresponding matrix H is

$$H=\begin{pmatrix} 1&1&1&1&1&1&1\\ 1&1&0&0&0&0&0\\ 1&0&1&0&1&1&0\\ 1&0&0&1&0&0&0\\ 1&0&1&0&1&1&0\\ 1&0&1&0&1&1&0\\ 1&0&0&0&0&0&1 \end{pmatrix}.$$

To check whether A has a positive semidefinite completion, we consider the problem

$$\begin{aligned} c^* = \ & \min \ ||H\circ(A-B)||_F^2\\ & \text{s.t.} \quad B \succcurlyeq 0 \end{aligned}$$

and its dual

$$\begin{aligned} d^* = \ & \max \ ||H\circ(A-B)||_F^2 - \operatorname{trace} \Lambda B\\ & \text{s.t.} \quad 2H\circ H\circ(B-A)=\Lambda\\ & \qquad\quad \Lambda \succcurlyeq 0, \end{aligned}$$

where $||\cdot||_F$ denotes the Frobenius matrix norm (see [36] for more details). Clearly if $c^*=0$, then the corresponding primal optimal solution B^* is an exact positive semidefinite completion of A. On the other hand, if we find a pair $(\bar{B},\bar{\Lambda})$ such that $||H\circ(A-\bar{B})||_F^2 - \operatorname{trace}\bar{\Lambda}\bar{B} > 0$, then because $c^* \geqslant d^*$ (by weak duality), it follows that $c^* > 0$ and hence A has no positive semidefinite completion.

Using this approach, we can find examples which prove that the inclusions in Corollary 3.2 are in fact strict for $n=5$, and hence for all $n \geqslant 5$.

3.1.2. Examples proving strict inclusions

In this section we prove that the inclusions in Corollary 3.2 are strict.

Example 3.1. Consider the matrix

$$X=\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}.$$

It is known that $X \notin C_5$ [42]. Applying the algorithm described in the previous section, we found a 16×16 matrix B^* which is feasible for SDP3 and such that $\mathrm{sMat}(B^*_{0,1:15}) = X$. The matrix B^* is defined as

$$B^*_{T(i,j),0} = \begin{cases} 1 & \text{if } i=j, \\ -\frac{1}{4} & \text{otherwise}, \end{cases}$$

$$B^*_{T(i,j),T(k,l)} = \begin{cases} 1 & \text{if } (i,j)=(k,l), \\ \frac{3}{8} & \text{if } (i,j) \text{ and } (k,l) \text{ are disjoint}, \\ -\frac{1}{4} & \text{otherwise}. \end{cases}$$

Hence, $X \in F_5$.

Example 3.2. Consider the matrix

$$X=\begin{pmatrix} 1 & -0.65 & -0.65 & -0.65 & 0.93 \\ -0.65 & 1 & 0.3 & 0.3 & -0.65 \\ -0.65 & 0.3 & 1 & 0.3 & -0.65 \\ -0.65 & 0.3 & 0.3 & 1 & -0.65 \\ 0.93 & -0.65 & -0.65 & -0.65 & 1 \end{pmatrix}.$$

It is easy to check that $X \in \mathscr{E}_5 \cap M_5$. Applying the algorithm described in the previous section, we found feasible matrices $\bar{B}$ and $\bar{\Lambda}$ for which the dual objective value is equal to $2.81 \times 10^{-4} > 0$. Hence, $c^* > 0$ and there is no matrix B feasible for SDP3 such that $\mathrm{sMat}(B_{0,1:15}) = X$. Hence, $X \notin F_5$. The matrices $\bar{B}$ and $\bar{\Lambda}$ can be found in the Technical Report [3] available on the Web site

`http://orion.math.uwaterloo.ca/~hwolkowi/henry/reports/ABSTRACTS.html`

Hence, we have proved that

Theorem 3.2. $C_n \subsetneq F_n \subsetneq \mathscr{E}_n \cap M_n$ *for* $n \geqslant 5$.

4. Geometry of the strengthened relaxations

We now study the geometrical structure of the feasible sets of our relaxations. In this section we focus solely on the relaxation SDP3 but all the results can be extended easily to SDP2.

Let $\mathscr{Z}$ be the set of $Z \in \mathscr{S}^{t(n)+1}$ that are feasible for SDP3. Since $\mathscr{Z}$ has no strictly feasible points, we seek to express $\mathscr{Z}$ in a lower dimensional space. We now show that this can be done without losing the sparsity of the constraints.

From the direct second lifting Section 2.2, we know that the 2^{n-1} matrices

$$Z_v := \begin{pmatrix} 1 \\ x_v \end{pmatrix} \begin{pmatrix} 1 \\ x_v \end{pmatrix}^{\mathrm{T}}, \quad x_v := \mathrm{svec}(vv^{\mathrm{T}}), \quad v \in \mathscr{V} := \{\pm 1\}^n$$

all belong to $\mathscr{Z}$. Furthermore, since these are all the points we are interested in, we want to project the feasible set onto $\mathscr{F}$, the minimal face of the positive semidefinite cone (in $\mathscr{S}^{t(n)+1}$) s.t. $Z_v \in \mathscr{F}\ \forall v \in \mathscr{V}$.

Consider the barycenter of the set of points Z_v:

$$\hat{Z} := 2^{-n} \sum_{v \in \mathscr{V}} Z_v.$$

By definition of $\mathscr{F}, \hat{Z} \in \mathrm{relint}\, \mathscr{F}$. Since $\mathscr{F}$ is a proper face, we can find a mapping from a lower dimensional positive semidefinite cone to $\mathscr{F}$. We construct this mapping using the results of the next theorem, which describes some of the structure of $\hat{Z}$.

Let $P_{i,j}$ denote the $(t(n)+1) \times (t(n)+1)$ permutation matrix equal to the identity matrix with the ith and jth columns permuted. We define the (permutation) matrix P as the following product of permutation matrices:

$$P := P_{2,t(2)} P_{3,t(3)} \dots P_{n,t(n)}.$$

Theorem 4.1. *The following statements hold for the barycenter* $\hat{Z}$:
1. $\hat{Z}$ *is a* $\{0,1\}$*-matrix and*

$$\hat{Z}_{ij} = \begin{cases} 1 & if\ i = t(k),\ j = t(l),\ k, l \in \{1, \dots, n\},\ k \neq l, \\ 1 & if\ i = j \in \{0, 1, \dots, t(n)\}, \\ 0 & elsewhere. \end{cases}$$

2. *The rank of* $\hat{Z}$ *is* $t(n-1)+1$ *and the eigenvalues are* $(n+1, 1, 0)$ *with multiplicities* $(1, t(n-1), n)$, *respectively.*
3. *The null space and range space of* $\hat{Z}$ *are*

$$\mathscr{N}(\hat{Z}) = \mathscr{R}\left(P \begin{bmatrix} V \\ 0 \end{bmatrix} \right)$$

and

$$\mathscr{R}(\hat{Z}) = \mathscr{R}\left(P \begin{bmatrix} e & 0 \\ 0 & I_{t(n-1)} \end{bmatrix} \right),$$

respectively, where $V \in \Re^{(n+1) \times n}$ *is any matrix s.t.* $[\, e \ \ V \,]$ *is an orthogonal matrix.*

Proof. 1. Let $v \in \mathscr{V}$ and consider Z_v. The elements of x_v have the form $(x_v)_j = v_\alpha v_\beta$, where $j = t(\beta - 1) + \alpha$ for $\alpha, \beta \in \{1, \dots, n\}$, $\alpha \neq \beta$, and furthermore

$$(x_v)_j = \begin{cases} 1 & \text{if } v_\alpha = v_\beta, \\ -1 & \text{otherwise.} \end{cases}$$

First consider the case when $\alpha=\beta=k$; here $j=t(k)$ and it is clear that $(x_v)_{t(k)}=1$, $k=1,\ldots,n$. This holds independently of the choice of v so we may conclude that

$$\begin{pmatrix} 1 \\ x_v \end{pmatrix}_{t(k)} = 1, \quad k=0,\ldots,n \quad \forall v \in \mathscr{V}. \tag{4.1}$$

Now suppose $\alpha \neq \beta$; then $v_\alpha = v_\beta$ for exactly 2^{n-1} elements of $\mathscr{V}$ and $v_\alpha \neq v_\beta$ for the other 2^{n-1} choices of v. Hence,

$$\sum_{v\in\mathscr{V}} (x_v)_j = 0 \quad \forall j \notin \{t(0),\ldots,t(n)\}. \tag{4.2}$$

Eqs. (4.1) and (4.2) together imply that the 0th column of $\hat{Z}$ equals $\sum_{k=0}^{n} e_{t(k)}$, i.e.

$$\hat{Z}_{i,0} = \begin{cases} 1 & \text{if } i \in \{t(0),\ldots,t(n)\}, \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry of $\hat{Z}$, $\hat{Z}_{0,j}=\hat{Z}_{j,0}$, so it remains to examine $\hat{Z}_{i,j}$ for $i,j=1,\ldots,t(n)$.

The remaining $t(n)$ columns of $\hat{Z}$ are

$$\hat{Z}_{:,j} = 2^{-n} \sum_{v\in\mathscr{V}} (x_v)_j \begin{pmatrix} 1 \\ x_v \end{pmatrix}$$

for $j=1,\ldots,t(n)$. If $i=j$ then $\hat{Z}_{i,i}=2^{-n}\sum_{v\in\mathscr{V}}(x_v)_i^2=1$, so we now suppose $i\neq j$.

If $i=t(k)$ and $j=t(l)$ for some $k,l\in\{1,\ldots,n\}$, $k\neq l$, then

$$\hat{Z}_{i,j} = 2^{-n} \sum_{v\in\mathscr{V}} (x_v)_{t(k)}(x_v)_{t(l)} = 1$$

using (4.1).

If $i\neq t(k)$, $\forall k$ but $j=t(l)$, then

$$\hat{Z}_{i,j} = 2^{-n} \sum_{v\in\mathscr{V}} (x_v)_i = 1$$

using (4.1) and (4.2). The case $i=t(k)$ but $j\neq t(l)$ $\forall l$ is handled similarly.

Finally, if $i\neq t(k)$ $\forall k$, and $j\neq t(l)$ $\forall l$, then we need only observe that $((x_v)_i,(x_v)_j)=(1,1)$ in exactly 2^{n-2} elements of $\mathscr{V}$, and the same count also holds for each of the combinations $(1,-1),(-1,1),(-1,-1)$. Thus, $\sum_{v\in\mathscr{V}}(x_v)_i(x_v)_j=0$. Hence $\hat{Z}_{i,j}=0$.

2. Define

$$\hat{Z}_P := P^{\mathrm{T}}\hat{Z}P = \begin{bmatrix} E & 0 \\ 0 & I_{t(n)-n} \end{bmatrix} \in \mathscr{S}^{(t(n)+1)\times(t(n)+1)}.$$

Since this is a similarity transformation, $\hat{Z}$ and $\hat{Z}_P$ have exactly the same eigenvalues and it suffices to prove the result for $\hat{Z}_P$. Also, $\hat{Z}_P$ is block diagonal, so its eigenvalues are those of the blocks. The lower block has the eigenvalue 1 with multiplicity $t(n)-n=t(n-1)$ (we have the set of standard eigenvectors $e_{n+1},\ldots,e_{t(n)}$). The upper block is clearly rank-1; since

$$\hat{Z}_P \begin{pmatrix} e \\ 0 \end{pmatrix} = (n+1) \begin{pmatrix} e \\ 0 \end{pmatrix},$$

$n + 1$ is its only non-zero eigenvalue. For V as in the statement of the theorem, $\hat{Z}_P\binom{V}{0} = 0$. So the columns of V (extended with zeros) give a set of eigenvectors for the zero eigenvalue, which has multiplicity n.

3. The result follows by the similarity of $\hat{Z}$ and $\hat{Z}_P$ and the proof of the previous part of the theorem. □

Now define the matrix

$$W := P \begin{bmatrix} e & 0 \\ 0 & I_{t(n-1)} \end{bmatrix} \in \Re^{(t(n)+1)\times(t(n-1)+1)}$$

with $e \in \Re^{n+1}$. Then $\mathscr{R}(\hat{Z}) = \mathscr{R}(W)$ and W provides a mapping from $\mathscr{S}^{t(n)+1}$ to the minimal face $\mathscr{F}$: if $Z \in \mathscr{F}$, $Z = WZ_PW^{\mathrm{T}}$ for $Z_P \in \mathscr{S}^{t(n-1)+1}$, and we require $Z_P \succcurlyeq 0$ to stay in the positive semidefinite cone of the lower dimensional space.

The projected version of SDP3 is thus

$$\begin{array}{rl} v_3^* = \max & \mathrm{trace}(W^{\mathrm{T}}H_QW)Z_P \\ \text{s.t.} & \mathrm{trace}(W^{\mathrm{T}}E_{ii}W)Z_P = 1, \quad i = 0,\ldots,t(n), \\ & \mathrm{trace}(W^{\mathrm{T}}E_{0,t(i)}W)Z_P = 1, \quad i = 1,\ldots,n, \\ & \mathrm{trace}(W^{\mathrm{T}}(E_{0,T(i,j)} - E_{T(i,k),T(k,j)})W)Z_P = 0 \forall k \quad \forall 1 \leqslant i < j \leqslant n, \\ & Z_P \succcurlyeq 0, \quad Z_P \in \mathscr{S}^{t(n-1)+1}, \end{array}$$

where $E_{ij} := \frac{1}{2}(e_ie_j^{\mathrm{T}} + e_je_i^{\mathrm{T}})$.

It remains to remove all the redundant constraints in this problem.

Let w_i denote the ith column of W^{T}. The construction of W implies $w_0^{\mathrm{T}} = w_{t(i)}^{\mathrm{T}} = e_0^{\mathrm{T}}$ $\forall i \in \{1,\ldots,n\}$, and the remaining columns of W:

$$\{w_{T(i,j)}^{\mathrm{T}} \colon i,j \in \{1,\ldots,n\}, i < j\} = \{e_1^{\mathrm{T}}, e_2^{\mathrm{T}}, \ldots, e_{t(n-1)}^{\mathrm{T}}\}$$

form a linearly independent set. (Together with w_0^{T}, they form a basis for $\Re^{t(n-1)+1}$.) Now, since $W^{\mathrm{T}}E_{ii}W = w_iw_i^{\mathrm{T}}$ and $W^{\mathrm{T}}E_{0,t(i)}W = \frac{1}{2}(w_0w_{t(i)}^{\mathrm{T}} + w_{t(i)}^{\mathrm{T}}w_0^{\mathrm{T}})$, we have

$$W^{\mathrm{T}}E_{t(i),t(i)}W = w_0w_0^{\mathrm{T}} = W^{\mathrm{T}}E_{00}W \quad \forall i \in \{1,\ldots,n\}$$

and

$$W^{\mathrm{T}}E_{0,t(i)}W = w_0w_0^{\mathrm{T}} = W^{\mathrm{T}}E_{00}W \quad \forall i \in \{1,\ldots,n\}.$$

Furthermore,

$$\begin{aligned} & W^{\mathrm{T}}(E_{0,T(i,j)} - E_{T(i,k),T(k,j)})W \\ & \quad = \tfrac{1}{2}\{w_0w_{T(i,j)}^{\mathrm{T}} + w_{T(i,j)}w_0^{\mathrm{T}} - w_{T(i,k)}w_{T(k,j)}^{\mathrm{T}} - w_{T(k,j)}w_{T(i,k)}^{\mathrm{T}}\}, \end{aligned}$$

therefore if $k = i$ or j then $w_{T(i,k)} = w_0$ or $w_{T(k,j)} = w_0$, respectively, so $W^{\mathrm{T}}(E_{0,T(i,j)} - E_{T(i,k),T(k,j)})W = 0$ and the corresponding constraint is redundant. Removing all these redundant constraints, we obtain $\mathrm{SDP3_P}$:

$$\begin{array}{lll}(\mathrm{SDP3_P})\, v_3^* = & \max & \operatorname{trace}(W^{\mathrm{T}} H_Q W) Z_P \\ & \text{s.t.} & \operatorname{trace}(W^{\mathrm{T}} E_{ii} W) Z_P = 1, \\ & & \quad i \in \{0, 1, \ldots, t(n)\} \setminus \{t(1), \ldots, t(n)\} \\ & & \operatorname{trace}(W^{\mathrm{T}}(E_{0,T(i,j)} - E_{T(i,k),T(k,j)}) W) Z_P = 0, \\ & & \quad \forall k \notin \{i, j\}, \quad \forall 1 \leqslant i < j \leqslant n \\ & & Z_P \succcurlyeq 0, \quad Z_P \in \mathscr{S}^{t(n-1)+1}.\end{array}$$

It is straightforward to check that all the remaining constraints are linearly independent. Moreover, we prove that Slater's constraint qualification holds for $\mathrm{SDP3_P}$. This implies that the optimal values of $\mathrm{SDP3_P}$ and its dual are equal and we can use a primal-dual interior-point algorithm.

First we simplify our notation. We have the following primal-dual pair:

$$\begin{array}{lll}(\mathrm{SDP3_P}) & \max & \operatorname{trace} CZ \\ & \text{s.t.} & \operatorname{diag} Z = e, \\ & & \operatorname{trace} A_{ijk} Z = 0 \quad \forall (i,j,k) \in \mathscr{J}, \\ & & Z \succcurlyeq 0, \quad Z \in \mathscr{S}^{t(n-1)+1}, \\ (\mathrm{DSDP3_P}) & \min & \displaystyle\sum_{i=1}^{t(n-1)+1} x_i \\ & \text{s.t.} & S = \operatorname{Diag}(x) + \displaystyle\sum_{(i,j,k) \in \mathscr{J}} y_{ijk} A_{ijk} - C, \\ & & S \succcurlyeq 0, \\ & & x \in \Re^{t(n-1)+1}, \quad y \in \Re^{(n-2)\cdot t(n-1)},\end{array}$$

where

$$\mathscr{J} := \{(i,j,k) \colon i,j \in \{1, \ldots, n\}, i < j, k \notin \{i,j\}\},$$

$$A_{ijk} := W^{\mathrm{T}}(E_{0,T(i,j)} - E_{T(i,k),T(k,j)}) W \quad \forall (i,j,k) \in \mathscr{J}$$

and

$$C := W^{\mathrm{T}} H_Q W.$$

Lemma 4.1. *Slater's constraint qualification holds for* $\mathrm{SDP3_P}$.

Proof. We consider the matrix $\tilde{Z} := I_{t(n-1)+1}$. Since $\tilde{Z} \succ 0$, we only need to verify that it satisfies the equality constraints.

Clearly, $\operatorname{diag} \tilde{Z} = e$. Now observe that

$$\begin{aligned} WW^{\mathrm{T}} &= P \begin{bmatrix} e & 0 \\ 0 & I_{t(n-1)} \end{bmatrix} \begin{bmatrix} e^{\mathrm{T}} & 0 \\ 0 & I_{t(n-1)} \end{bmatrix} P^{\mathrm{T}} \\ &= P \begin{bmatrix} E & 0 \\ 0 & I_{t(n-1)} \end{bmatrix} P^{\mathrm{T}} \end{aligned}$$

$$= P\hat{Z}_P P^{\mathrm{T}}$$
$$= \hat{Z},$$

where $\hat{Z}_P$ is the matrix defined in the proof of Theorem 4.1.

Using this observation, the second set of equality constraints for $\tilde{Z}$ may be written as

$$\operatorname{trace}(E_{0,T(i,j)} - E_{T(i,k),T(k,j)})\hat{Z} = 0 \quad \forall (i,j,k) \in \mathscr{J}$$

and these equalities hold because

$$\operatorname{trace}(E_{0,T(i,j)} - E_{T(i,k),T(k,j)})\hat{Z} = 0 \quad \Leftrightarrow \quad \hat{Z}_{0,T(i,j)} = \hat{Z}_{T(i,k),T(k,j)}$$

and by Theorem 4.1(1) both entries of $\hat{Z}$ involved are zero. □

It is straightforward to prove that the same is true for the dual problem.

Lemma 4.2. *Slater's constraint qualification holds for* DSDP3$_{\mathrm{P}}$.

Proof. Choosing $\tilde{y}_{ijk} := 0 \ \forall (i,j,k) \in \mathscr{J}$ and $\tilde{x}_i := ||\mathrm{dsvec}(Q)||_1 + 1 \ \forall i = 1,\ldots,t(n-1)+1$, the corresponding dual (slack) variable is

$$\tilde{S} = (||\mathrm{dsvec}(Q)||_1 + 1)I_{t(n-1)+1} - C$$

which is strictly diagonally dominant and has all its diagonal entries positive. Hence $\tilde{S}$ is positive definite. □

All the results in this section extend to the relaxation SDP2. The corresponding projected problem is

$$\begin{array}{llll} (\mathrm{SDP2_P}) & v_2^* = & \max & \operatorname{trace} CY_P \\ & & \text{s.t.} & \operatorname{diag} Y_P = e, \\ & & & \operatorname{trace}(W^{\mathrm{T}} R_{ij} W) Y_P = 0 \quad \forall 1 \leqslant i < j \leqslant n, \\ & & & Y_P \succcurlyeq 0, \quad Y_P \in \mathscr{S}^{t(n-1)+1}, \end{array}$$

where

$$R_{ij} := nE_{0,T(i,j)} - \sum_{k=1}^{n} E_{T(i,k),T(k,j)}.$$

5. Numerical comparison of the relaxations

The relaxations SDP1, SDP2$_{\mathrm{P}}$ and SDP3$_{\mathrm{P}}$ were compared for several interesting problems using the software package SDP$_{\mathrm{Pack}}$ (version 0.9 Beta) [1]. For completeness we also solved the metric polytope relaxation:

$$\begin{array}{ll} \max & \operatorname{trace} QX \\ \text{s.t.} & X \in M_n. \end{array}$$

This relaxation is easily formulated as an LP and we solved it using the Matlab solver LINPROG. The results are summarized in Table 1. A relative error equal to zero means

Table 1
Numerical comparison of all MC relaxations for selected test problems

Graph	μ^*	SDP1 bound	SDP2P bound	M_n bound	$\mathcal{E}_n \cap M_n$ bound	SDP3P bound
C_5	4	4.5225 $\rho = 0.8845$ R.E.: 13.06%	4.2889 $\rho = 0.9326$ R.E.: 7.22%	4.0000 $\rho = 1.0000$ R.E.: 0%	4.0000 $\rho = 1.0000$ R.E.: 0%	4.0000 $\rho = 1.0000$ R.E.: 0%
$K_5 \setminus e$	6	6.2500 $\rho = 0.9600$ R.E.: 4.17%	6.1160 $\rho = 0.9810$ R.E.: 1.93%	6.0000 $\rho = 1.0000$ R.E.: 0%	6.0000 $\rho = 1.0000$ R.E.: 0%	6.0000 $\rho = 1.0000$ R.E.: 0%
K_5	6	6.2500 $\rho = 0.9600$ R.E.: 4.17%	6.2500 $\rho = 0.9600$ R.E.: 4.17%	6.6667 $\rho = 0.9000$ R.E.: 11.11%	6.2500 $\rho = 0.9600$ R.E.: 4.17%	6.2500 $\rho = 0.9600$ R.E.: 4.17%
Given by $A(G)$ $(n = 5)$	9.28	9.6040 $\rho = 0.9663$ R.E.: 3.49%	9.4056 $\rho = 0.9866$ R.E.: 1.35%	9.3867 $\rho = 0.9886$ R.E.: 1.15%	9.2961 $\rho = 0.9983$ R.E.: 0.17%	9.2800 $\rho = 1.0000$ R.E.: 0%
AW_9^2	12	13.5 $\rho = 0.8889$ R.E.: 12.50%	12.9827 $\rho = 0.9243$ R.E.: 8.19%	12.8571 $\rho = 0.9333$ R.E.: 7.14%	12.6114 $\rho = 0.9515$ R.E.: 5.10%	12.4967 $\rho = 0.9603$ R.E.: 4.14%
Pet. $(n = 10)$	12	12.5 $\rho = 0.9600$ R.E.: 4.17%	12.3781 $\rho = 0.9695$ R.E.: 3.15%	12.0000 $\rho = 1.0000$ R.E.: 0%	12.0000 $\rho = 1.0000$ R.E.: 0%	12.0000 $\rho = 1.0000$ R.E.: 0%
Given in [3] $(n = 12)$	88	90.3919 $\rho = 0.9735$ R.E.: 2.72%	89.5733 $\rho = 0.9824$ R.E.: 1.79%	89.3333 $\rho = 0.9851$ R.E.: 1.52%	88.0029 $\rho = 1.0000$ R.E.: $3.3E - 5$	88.0000 $\rho = 1.0000$ R.E.: $9.9E - 7$

that the relative error was below 10^{-11}. The value ρ equals the value of the optimal cut divided by the bound, and R.E. denotes the relative error with respect to the optimal cut.

The test problems in Table 1 are as follows:

1. The first line of results corresponds to solving the three SDP relaxations for a 5-cycle with unit edge-weights.
2. The second line corresponds to the complete graph on 5 vertices with unit edge-weights on all edges except one, which is assigned weight zero.
3. The third line corresponds to the complete graph on 5 vertices with unit edge-weights. In this example, none of the four SDP relaxations attains the MC optimal value, and in fact they are not numerically distinguishable. Only the polyhedral relaxation M_n gives a noticeably weaker bound.
4. The fourth line corresponds to the graph defined by the weighted adjacency matrix

$$A(G) = \begin{pmatrix} 0 & 1.52 & 1.52 & 1.52 & 0.16 \\ 1.52 & 0 & 1.60 & 1.60 & 1.52 \\ 1.52 & 1.60 & 0 & 1.60 & 1.52 \\ 1.52 & 1.60 & 1.60 & 0 & 1.52 \\ 0.16 & 1.52 & 1.52 & 1.52 & 0 \end{pmatrix}.$$

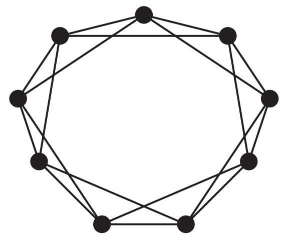

Fig. 2. Antiweb AW_9^2.

This problem is interesting because it shows a significant difference between $SDP3_P$ and all the other relaxations; in this case, $SDP3_P$ is the only relaxation that attains the MC optimal value.

5. The fifth line corresponds to the graph in Fig. 2 with unit edge weights. [5] This graph is the antiweb AW_9^2 and it is interesting that on this example, unlike for the K_5 with unit edge weights, $SDP3_P$ performs better than the SDP1 relaxation with all the triangle inequalities included.
6. The last two lines correspond to slightly larger graphs. The first one has 10 vertices; it is the Petersen graph with unit edge-weights. The second one is a graph with 12 vertices that gives slightly different results for each relaxation; its weighted adjacency matrix can be found in [3].

The numerical results of Table 1 cover only small problems. This is because solving the relaxations $SDP2_P$ and $SDP3_P$ using SDPpack becomes extremely time-consuming and requires large amounts of memory even for moderate values of n. To verify the behaviour of the relaxations on larger problems, we considered two other SDP packages. One was CSDP (version 2.3), a C implementation of an interior-point method developed by Borchers [13,14] and accessible via the NEOS Server for Optimization at `http://www-neos.mcs.anl.gov`. The second package was SBmethod, a C++ implementation of the spectral bundle method developed by Helmberg [32,29,28]. Using these packages, we obtained the results presented in Table 2.

The bound SDP1 is known to be excellent both theoretically ($\rho \geqslant 0.878$ [23]) and empirically ($\rho \cong 0.97$, see e.g. [33]). Nonetheless, our numerical experiments to date suggest that $SDP2_P$ and $SDP3_P$ consistently yield a *strict* improvement over SDP1, and that on randomly generated test problems (with non-negative integer weights),

the $SDP3_P$ relaxation often yields the *optimal* value of MC.

It is also important to note that the constraints of both relaxations are sparse and their sparsity increases rapidly with n. This special structure has not yet been exploited.

Finally, we note that the direct lifting approach in Section 2.2 shows that the strengthened relaxations SDP2 and SDP3 also provide bounds for quartic Max-Cut

[5] Thanks to Franz Rendl for suggesting this example.

Table 2
Numerical comparison of SDP1, SDP2_P and SDP3_P on randomly generated graphs with non-negative edge weights

Number of vertices	μ^*	SDP1 bound	SDP2_P bound	SDP3_P bound
10	648	666.428 $\rho = 0.9723$ R.E.: 2.84%	656.8020 $\rho = 0.9866$ R.E.: 1.36%	648.000 $\rho = 1.0000$ R.E.: 0%
11	1060	1084.345 $\rho = 0.9775$ R.E.: 2.30%	1072.352 $\rho = 0.9885$ R.E.: 1.17%	1060.000 $\rho = 1.0000$ R.E.: 0%
15	2290	2317.354 $\rho = 0.9882$ R.E.: 1.19%	2301.634 $\rho = 0.9949$ R.E.: 0.51%	2290.000 $\rho = 1.0000$ R.E.: 0%
16	2270	2318.867 $\rho = 0.9789$ R.E.: 2.15%	2300.354 $\rho = 0.9868$ R.E.: 1.34%	2270.000 $\rho = 1.0000$ R.E.: 0%
25	380	385.4737 $\rho = 0.9858$ R.E.: 1.44%	383.6503 $\rho = 0.9905$ R.E.: 0.96%	380.000 $\rho = 1.0000$ R.E.: 0%
30	1705.5	1751.600 $\rho = 0.9737$ R.E.: 2.70%	1743.205 $\rho = 0.9784$ R.E.: 2.21%	1705.578 $\rho = 1.0000$ R.E.: $4.6E-5$
33	1888.5	1932.968 $\rho = 0.9770$ R.E.: 2.35%	1926.119 $\rho = 0.9805$ R.E.: 1.99%	1888.564 $\rho = 1.0000$ R.E.: $3.4E-5$
36	27108.55	28305.28 $\rho = 0.9577$ R.E.: 4.41%	27944.30 $\rho = 0.9701$ R.E.: 3.08%	27108.81 $\rho = 1.0000$ R.E.: $9.8E-6$

problems, i.e. problems of the form:

$$\max \sum_{1 \leqslant i<j<k<l \leqslant n} H_{T(i,j),T(k,l)}\, v_i v_j v_k v_l + \sum_{1 \leqslant i<j \leqslant n} H_{0,T(i,j)}\, v_i v_j$$

$$\text{s.t.} \quad v_i^2 = 1, \quad i = 1,\ldots,n.$$

Further research on such problems may prove the computational effort involved in solving the relaxations to be worthwhile.

6. Conclusion

We have presented two strengthened semidefinite programming relaxations for the Max-Cut problem and proved several interesting properties of these relaxations. In particular, we proved that the tighter of these two relaxations corresponds to a relaxation of the cut polytope that is strictly contained in the intersection of the elliptope and the metric polytope. Our results illustrate the strength and flexibility of Lagrangian relaxation for obtaining a variety of SDP relaxations with different properties.

We also addressed some practical issues in the solution of these SDP relaxations. Preliminary numerical results show a strict improvement over the Goemans–Williamson relaxation, and show the tighter relaxation often yielding the optimal value of Max-Cut for randomly generated test problems. Although these relaxations have many variables and linear constraints, current research efforts promise to yield efficient methods for solving them.

Acknowledgements

We thank Michel X. Goemans for helpful comments that led to the proof of Theorem 3.1. We also thank an anonymous referee for pointing us to useful references and helping with the presentation of the material in this paper.

References

[1] F. Alizadeh, J.-P. Haeberly, M.V. Nayakkankuppam, M.L. Overton, S. Schmieta, SDP pack user's guide—version 0.9 Beta, Technical Report TR1997-737, Courant Institute of Mathematical Sciences, NYU, New York, NY, June 1997.

[2] M.F. Anjos, H. Wolkowicz, A strengthened SDP relaxation via a second lifting for the Max-Cut problem, Technical Research Report, CORR 99-55, University of Waterloo, Waterloo, Ont., 1999, 28p.

[3] M.F. Anjos, H. Wolkowicz, A tight semidefinite relaxation of the cut polytope, Technical Research Report, CORR 2000-19, University of Waterloo, Waterloo, Ont., 2000, 24p.

[4] K.M. Anstreicher, X. Chen, H. Wolkowicz, Y. Yuan, Strong duality for a trust-region type relaxation of the quadratic assignment problems, Linear Algebra Appl. 301 (1–3) (1999) 121–136.

[5] K.M. Anstreicher, H. Wolkowicz, On Lagrangian relaxation of quadratic matrix constraints, SIAM J. Matrix Anal. Appl. 22 (1) (2000) 41–55.

[6] E. Balas, A modified lift-and-project procedure, Math. Programming Ser. B 79(1–3) 19–31, 1997, Lectures on Mathematical Programming, ismp97, Lausanne, 1997.

[7] E. Balas, S. Ceria, G. Cornuejols, A lift-and-project cutting plane algorithm for mixed 0-1 programs, Math. Programming 58 (1993) 295–324.

[8] E. Balas, S. Ceria, G. Cornuéjols, Solving mixed 0-1 programs by a lift-and-project method, in: Proceedings of the Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, Austin, TX, 1993, ACM, New York, 1993, pp. 232–242.

[9] F. Barahona, The max-cut problem on graphs not contractible to K_5, Oper. Res. Lett. 2 (3) (1983) 107–111.

[10] F. Barahona, On cuts and matchings in planar graphs, Math. Programming Ser. A 60 (1) (1993) 53–68.

[11] F. Barahona, M. Grötschel, M. Jünger, G. Reinelt, An application of combinatorial optimization to statistical physics and circuit layout design, Oper. Res. 36 (1988) 493–513.

[12] S.J. Benson, Y. Ye, X. Zhang, Solving large-scale sparse semidefinite programs for combinatorial optimization, SIAM J. Optim. 10(2) (2000) 443–461 (electronic).

[13] B. Borchers, CSDP, a C library for semidefinite programming, Optim. Methods Software (Interior point methods) 11/12 (1–4) (1999) 613–623.

[14] B. Borchers, SDPLIB 1.2, a library of semidefinite programming test problems, Optim. Methods Software (Interior point methods) 11 (1) (1999) 683–690.

[15] S. Burer, R.D.C. Monteiro, An efficient algorithm for solving the MAXCUT SDP relaxation, Technical Report, Georgia Tech., Atlanta, GA, 1999.

[16] C. Delorme, S. Poljak, Laplacian eigenvalues and the maximum cut problem, Math. Programming 62 (3) (1993) 557–574.
[17] M.M. Deza, M. Laurent, Geometry of Cuts and Metrics, Springer, Berlin, 1997.
[18] U. Feige, G. Schechtman, On the optimality of the random hyperplane rounding technique for MAX CUT, Technical Report, Weizmann Institute, Rehovot, Israel, 2000.
[19] M.R. Garey, D.S. Johnson, Computers and Intractabilty: A Guide to the Theory of NP-Completeness, Freeman, San Francisco, 1979.
[20] M.X. Goemans, Semidefinite programming in combinatorial optimization, Math. Programming 79 (1997) 143–162.
[21] M.X. Goemans, Semidefinite programming and combinatorial optimization, Documenta Math. ICM 1998 (1998) 657–666 (Invited talk at the International Congress of Mathematicians, Berlin, 1998).
[22] M.X. Goemans, F. Rendl, Combinatorial optimization, in: H. Wolkowicz, R. Saigal, L. Vandenberghe (Eds.), Handbook of Semidefinite Programming: Theory, Algorithms, and Applications, Kluwer Academic Publishers, Boston, MA, 2000.
[23] M.X. Goemans, D.P. Williamson, .878-approximation algorithms for MAX CUT and MAX 2SAT, in: ACM Symposium on Theory of Computing (STOC), Montréal, Québec, 1994.
[24] M.X. Goemans, D.P. Williamson, Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, J. Assoc. Comput. Mach. 42 (6) (1995) 1115–1145.
[25] B. Grone, C.R. Johnson, E. Marques de Sa, H. Wolkowicz, Positive definite completions of partial Hermitian matrices, Linear Algebra Appl. 58 (1984) 109–124.
[26] J. Hastad, Some optimal inapproximability results, in: Proceedings of the 29th ACM Symposium on Theory and Computation, 1997.
[27] C. Helmberg, An interior-point method for semidefinite programming and max-cut bounds, PhD Thesis, Graz University of Technology, Austria, 1994.
[28] C. Helmberg, SBmethod—A C++ implementation of the spectral bundle method, ZIB preprint 00-35, Konrad-Zuse-Zentrum für Informationstechnik Berlin, Takustraße 7, 14196, Berlin, Germany, October 2000.
[29] C. Helmberg, K.C. Kiwiel, A spectral bundle method with bounds, ZIP preprint sc-99-37, Konrad-Zuse-Zentrum für Informationstechnik Berlin, Takustraße 7, 14195, Berlin, Germany, November 1999.
[30] C. Helmberg, F. Oustry, Bundle methods to minimize the maximum eigenvalue function, in: H. Wolkowicz, R. Saigal, L. Vandenberghe (Eds.), Handbook of Semidefinite Programming: Theory, Algorithms, and Applications, Kluwer Academic Publishers, Boston, MA, 2000.
[31] C. Helmberg, S. Poljak, F, Rendl, H. Wolkowicz, Combining semidefinite and polyhedral relaxations for integer programs, in: Integer Programming and Combinatorial Optimization (Copenhagen, 1995), Springer, Berlin, 1995, pp. 124–134.
[32] C. Helmberg, F. Rendl, A spectral bundle method for semidefinite programming, SIAM J. Optim. 10 (3) (2000) 673–696.
[33] C. Helmberg, F. Rendl, R.J. Vanderbei, H. Wolkowicz, An interior-point method for semidefinite programming, SIAM J. Optim. 6 (2) (1996) 342–361.
[34] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1994 (corrected reprint of the 1991 original).
[35] C.R. Johnson, Matrix completion problems: a survey, Proc. Symp. Appl. Math. 40 (1990) 171–198.
[36] C.R. Johnson, B. Kroschel, H. Wolkowciz, An interior-point method for approximate positive semidefinite completions, Comput. Optim. Appl. 9 (2) (1998) 175–190.
[37] R.M. Karp, Reducibility among combinatorial problems, in: R.E. Miller, J.W. Thatcher (Eds.), Complexity of Computer Computation, Plenum Press, New York, 1972, pp. 85–103.
[38] J.B. Lasserre, Optimality conditions and LMI relaxations for 0-1 programs, LAAS Research Report, LAAS-CNRS, Toulouse, France, 2000.
[39] M. Laurent, A tour d'horizon on positive semidefinite and Euclidean distance matrix completion problems, in: Topics in Semidefinite and Interior-Point Methods, The Fields Institute for Research in Mathematical Sciences, Communications Series, Vol. 18, American Mathematical Society, Providence, RI, 1998, pp. 51–76.
[40] M. Laurent, S. Poljak, On a positive semidefinite relaxation of the cut polytope, Linear Algebra Appl. 223/224 (1995) 439–461.

[41] M. Laurent, S. Poljak, On the facial structure of the correlation matrices, SIAM J. Matrix Anal. Appl. 17 (3) (1996) 530–547.
[42] M. Laurent, S. Poljak, F. Rendl, Connections between semidefinite relaxations of the max-cut and stable set problems, Math. Programming 77 (1997) 225–246.
[43] T. Lengauer, Combinatorial Algorithms for Integrated Circuit Layout, Wiley, Chichester, 1990 (with a foreword by Bryan Preas).
[44] L. Lovász, A. Schrijver, Cones of matrices and set-functions and 0-1 optimization, SIAM J. Optim. 1 (2) (1991) 166–190.
[45] B. Mohar, S. Poljak, Eigenvalues in combinatorial optimization, in: Combinatorial Graph-Theoretical Problems in Linear algebra, IMA, Vol. 50, Springer, Berlin, 1993.
[46] Y.E. Nesterov, Quality of semidefinite relaxation for nonconvex quadratic optimization, Technical Report, CORE, Universite Catholique de Louvain, Belgium, 1997.
[47] Y.E. Nesterov, H. Wolkowicz, Y. Ye, Semidefinite programming relaxations of nonconvex quadratic optimization, in: H. Wolkowicz, R. Saigal, L. Vandenberghe (Eds.), Handbook of Semidefinite Programming: Theory, Algorithms and Applications, Kluwer Academic Publishers, Boston, MA, 2000, p. 34.
[48] S. Poljak, F, Rendl, H. Wolkowicz, A recipe for semidefinite relaxation for (0,1)-quadratic programming, J. Global Optim. 7 (1) (1995) 51–73.
[49] F. Rendl, Semidefinite programming and combinatorial optimization, Appl. Numer. Math. 29 (1999) 255–281.
[50] H.D. Sherali, W.P. Adams, A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems, SIAM J. Discrete Math. 3 (3) (1990) 411–430.
[51] H.D. Sherali, W.P. Adams, A hierarchy of relaxations and convex hull characterizations for mixed-integer zero-one programming problems, Discrete Appl. Math. 52 (1) (1994) 83–106.
[52] N.Z. Shor, Quadratic optimization problems, Izv. Akad. Nauk SSSR Tekhn. Kibernet. 222 (1) (1987) 128–139, 222.
[53] R. Stern, H. Wolkowicz, Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations, SIAM J. Optim. 5 (2) (1995) 286–313.
[54] V.A. Yakubovich, The *S*-procedure and duality theorems for non-convex problems of quadratic programming, Vestnik Leningrad Univ. 1973 (1) (1973) 81–87.
[55] V.A. Yakubovich, Nonconvex optimization problem: the infinite-horizon linear-quadratic control problem with quadratic constraints, Systems Control Lett. 19 (1) (1992) 13–22.
[56] Y. Ye, Approximating quadratic programming with bound and quadratic constraints, Math. Programming 84 (1999) 219–226.