

Singularity degree of non-facially exposed faces

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Abstract

In this paper, we study the facial structure of the linear image of a cone. We define the singularity degree of a face of a cone to be the minimum number of steps it takes to expose it using exposing vectors from the dual cone. We show that the singularity degree of the linear image of a cone is exactly the number of facial reduction steps to obtain the minimal face in a corresponding primal conic optimization problem. This result generalizes the relationship between the complexity of general facial reduction algorithms and facial exposedness of conic images under a linear transform by Drusvyatskiy, Pataki and Wolkowicz to arbitrary singularity degree. We present our results in the original form and also in its nullspace form. As a by-product, we show that frameworks underlying a chordal graph have at most one level of stress matrix.

Keywords: Singularity degree, exposed face, convex cones, semidefinite programming

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1 Introduction

We consider the following linear *conic optimization problem* in its primal form,

$$(P) \quad \begin{array}{ll} \max & \langle C, X \rangle \\ \text{s.t.} & \mathcal{M}(X) = b \in \mathbb{F} \\ & X \in \mathcal{K} \subseteq \mathbb{E}, \end{array} \quad (1.1)$$

where $\mathcal{M} : \mathbb{E} \rightarrow \mathbb{F}$ is a surjective linear transformation between finite dimensional Euclidean spaces, $\langle \cdot, \cdot \rangle$ denotes the respective inner product, and $\mathcal{K} \subset \mathbb{E}$ is a convex cone.

The *feasible region*, \mathcal{F} , of this conic linear optimization problem is denoted

$$\mathcal{F} = \{X \in \mathcal{K} : \mathcal{M}(X) = b\}. \quad (1.2)$$

Understanding the structure of the image of the convex cone \mathcal{K} under a linear transformation is an important problem in optimization. Since \mathcal{M} is surjective and hence an *open mapping*, strict feasibility (also called Slater’s constraint qualification, $\exists \hat{X} \in \text{int } \mathcal{K}, \mathcal{M}(\hat{X}) = b$) requires that $b \in \text{int } \mathcal{M}(\mathcal{K})$. In addition, the linear image of a convex cone can have very different structure from the convex cone itself. The linear image of a closed convex cone is not necessarily closed, see [16] for characterizations. The convex cone \mathcal{K} can be *facially exposed*, i.e., any face f of \mathcal{K} can be written as $f = \mathcal{K} \cap v^\perp \trianglelefteq \mathcal{K}$ with v in the *dual cone*, \mathcal{K}^* , of \mathcal{K} , while simultaneously the linear image of \mathcal{K} is *not* facially exposed.

The smallest number of steps of facial reduction required to obtain the minimum face is called the *singularity degree* of the system (1.2) over \mathcal{K} . The singularity degree is an intriguing measurement of complexity for \mathcal{F} , see e.g., the case when \mathcal{K} is the cone of positive semidefinite matrices [20, 21]. Motivated by these results, in this paper we aim to generalize the notion of singularity degree to the case of the linear image of \mathcal{K} and in particular, for the case of not facially-exposed cones. We include results on extending the coordinate shadow theorem [9] of singularity degree one to higher singularity degree. Therefore we obtain an alternative characterization of the singularity degree of a linear optimization problem. We are also motivated by the work in

[7] on the universal rigidity of frameworks where the connections between facial reduction and stress matrices are discussed.

For a set S , we let $\text{face}(S, \mathcal{K})$ denote the smallest face of \mathcal{K} containing the set S . We note the following related result, reworded in our notation.

Theorem 1.1. [9, Theorem 4.1] *Let \mathcal{K} be closed. Then for any vector v exposing $\text{face}(b, \mathcal{M}(\mathcal{K}))$, the vector \mathcal{M}^*v exposes $\text{face}(\mathcal{F}, \mathcal{K})$.*

As mentioned in [9], “there is difficulty when working with $\mathcal{M}(K)$ as it is not usually a well-understood set; and systematically recognizing points on its boundary is hopeless and exposing vectors are out of reach.” This is the motivation for our study in this paper.

2 Background: facial structure of a convex cone

We begin with notation and then present some preliminary results.

2.1 Notation

Consider two Euclidean spaces \mathbb{E} and \mathbb{F} with inner products $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{F}}$, respectively. We use $\langle \cdot, \cdot \rangle$ when the meaning is clear. Define $\mathcal{M} : \mathbb{E} \rightarrow \mathbb{F}$ to be a linear surjective mapping between the Euclidean spaces \mathbb{E} and \mathbb{F} . The *adjoint of \mathcal{M}* , \mathcal{M}^* , is defined as the unique linear mapping from \mathbb{F} to \mathbb{E} such that $\langle \mathcal{M}(x), y \rangle_{\mathbb{F}} = \langle x, \mathcal{M}^*(y) \rangle_{\mathbb{E}}, \forall x \in \mathbb{E}, \forall y \in \mathbb{F}$. Consider a convex cone $\mathcal{K} \subset \mathbb{E}$. We denote $\text{int}(\mathcal{K}), \text{ri}(\mathcal{K})$ to be the interior and relative interior of \mathcal{K} , respectively; and we denote the orthogonal complement of \mathcal{K} as \mathcal{K}^{\perp} . For a vector v , we let $v^{\perp} := \{v\}^{\perp}$. We associate to \mathcal{K} its *dual cone*

$$\mathcal{K}^* = \{y \in \mathbb{E} : \langle y, x \rangle \geq 0, \forall x \in \mathcal{K}\}.$$

A convex subset $f \subseteq \mathcal{K}$ is called a *face* of \mathcal{K} , denoted $f \trianglelefteq \mathcal{K}$, if

$$x, y \in \mathcal{K}, \frac{1}{2}(x + y) \in f \implies x \in f, y \in f.$$

In other words, a face f contains all line segments in \mathcal{K} whose relative interior intersects f . The *minimal face* containing a set $S \in \mathcal{K}$, denoted $\text{face}(S, \mathcal{K})$, is the intersection of all faces of \mathcal{K} containing S . A face f of \mathcal{K} is an *exposed face* when there exists a vector $v \in \mathcal{K}^*$ such that $f = \mathcal{K} \cap v^{\perp}$. In this case, we say v exposes f . The cone \mathcal{K} is called *facially exposed* if all faces of \mathcal{K} are exposed. Examples of facially exposed cones includes: the nonnegative orthant \mathbb{R}_+^n ; the cone of positive semidefinite matrices \mathbb{S}_+^n ; and the cone of Euclidean distance matrices \mathcal{E}^n . The linear image of the (non-polyhedral) cones $\mathbb{S}_+^n, \mathcal{E}^n$ are not necessarily facially exposed, see [9].

2.1.1 SDP, EDM cones

We denote \mathcal{S}^n as the vector space of $n \times n$ real symmetric matrices. The inner product of two symmetric matrices is defined to be the trace $\langle A, B \rangle = \text{trace}(AB)$. The cones of positive semidefinite and positive definite matrices in \mathcal{S}^n are denoted as \mathbb{S}_+^n and \mathbb{S}_{++}^n , respectively. They give rise to the Löwner partial order

$$X \succeq Y \iff Y - X \in \mathbb{S}_+^n, \quad X \succ Y \iff Y - X \in \mathbb{S}_{++}^n.$$

For a general cone, we use $X \succeq_{\mathcal{K}} Y, X \succ_{\mathcal{K}} Y$. The cone of positive semidefinite matrices is facially exposed. Let $X \in \text{relint } f, f \trianglelefteq \mathbb{S}_+^n$, and $X = UDU^T$ be its compact (orthogonal) spectral

decomposition. Then, see e.g., [10],

$$f = \left\{ U \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} U^T : R \in \mathcal{S}_+^r \right\} = \mathbb{S}_+^n \cap (VV^T)^\perp,$$

where U is an orthogonal matrix and $r = \text{rank}(X)$. Here VV^T is the exposing vector for the face, i.e., the columns of V form a basis for the orthogonal complement of the range of U .

A matrix $D \in \mathcal{S}^n$ is called an *Euclidean distance matrix* (EDM for short) if there exists n points $p_i, i = 1, \dots, n$ in \mathbb{R}^k such that $D_{ij} = \|p_i - p_j\|^2$ for all indices i, j . We denote \mathcal{E}^n as the cone of all $n \times n$ Euclidean distance matrices. The EDM cone \mathcal{E}^n is linearly isomorphic to \mathcal{S}_+^{n-1} . See e.g., the book [2]. The EDM matrices and PSD matrices (Gram matrices) are connected by the *Lindenstrauss mapping*, K . There is an one-one correspondence between the centered PSD matrices and EDM matrices by Lindenstrauss mapping. We state the result as follows,

$$K := \mathcal{S}^n \rightarrow \mathcal{S}^n, K(X)_{ij} := X_{ii} + X_{jj} - 2X_{ij}.$$

with adjoint and *Moore-Penrose pseudoinverse*,

$$K^*(D) = 2(\text{Diag}(De) - D), \quad K^\dagger(D) = -\frac{1}{2}J_n \text{offDiag}(D)J_n,$$

respectively. Here Diag is the adjoint of diag , the matrix $J_n := I - \frac{1}{n}ee^T$ is the orthogonal projection onto SC, and $\text{offDiag}(D)$ refers to zeroing out the diagonal of D .

2.2 Facial reduction

In [9], the facial exposedness of the linear image of \mathcal{K} is shown to be equivalent to the singularity degree, [20, 21], of (1.1) being one. Note that the singularity degree is the minimum number of steps needed when applying a general facial reduction algorithm, [5], on a conic optimization problem.

A facial reduction algorithm aims to resolve lack of *strict feasibility* in the system (1.1). The lack of strict feasibility can cause many problems in an optimization problem. For example, one can have a positive duality gap, strong duality can fail to hold, and stability and numerical issues can arise. The facial reduction algorithm is motivated by the following *theorem of alternative* in conic optimization, see e.g., [5]:

Lemma 2.1. *Exactly one of the following two alternatives hold:*

1. *Strict feasibility holds in (1.2) ($\exists \hat{X} \in \text{ri}(\mathcal{K})$ such that $\mathcal{M}(\hat{X}) = b$).*
2. *There exists $v \in \mathbb{F}$ such that:*

$$\mathcal{K}^\perp \not\subseteq Z = \mathcal{M}^*v \in \mathcal{K}^*, \text{ and } \langle v, b \rangle = 0. \quad (2.1)$$

We call $Z \in \mathcal{K}^*$ an *exposing vector* for the face $f \trianglelefteq K$ if $X \in f \implies \langle Z, X \rangle = 0$. We see that the so-called *auxiliary system* (2.1) is equivalent to the existence of the special exposing vector Z . One can then replace \mathcal{K} by the *exposed face* $\mathcal{K} \cap Z^\perp \trianglelefteq \mathcal{K}$ and obtain a smaller problem with a smaller cone. This also results in a reduction of the number of constraints in (1.2) to maintain surjectivity, see [5]. This procedure can be repeated until strict feasibility is attained. We summarize the facial reduction algorithm over a convex cone in Algorithm 1, see [5, 10]:

Recall that the smallest number of steps of facial reduction required to obtain the minimum face is called the *singularity degree* of the system (1.2) over \mathcal{K} .

Algorithm 1 facial reduction

Input: (P)

Output: A sequence of reducing certificates Z_i , $i = 0, 1, \dots, k$

Set $i \leftarrow 0$, let $\mathcal{F}_i \leftarrow \mathcal{K}$, $i = 0, 1, \dots, k$,

Step 1: Find an exposing vector $Z_i = \mathcal{M}^*v_i \in (\mathcal{F}_i)^*$, $Z_i \notin \mathcal{F}_i^\perp$, $\langle v_i, b \rangle = 0$.

If no such Z_i exists, then exit.

Set $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap Z_i^\perp$

$i = i + 1$

Return to Step 1.

Output: the minimal face \mathcal{F}_k of \mathcal{K} containing the feasible region of (P).

The SDP cones and EDM cones are “nice” cones [16]. A closed convex cone \mathcal{K} is nice if $\mathcal{K}^* + E^\perp$ is closed for any face E of \mathcal{K} . Due to this “niceness”, one can construct a family of extended duals such that strong duality holds by the facial reduction procedure [17–19]. The length of the sequence in the extended duals is exactly the singularity degree.

Two closely related problems in the literature are the EDM completion problem and PSD completion problem. In those problems, we are given a partial EDM or PSD matrix whose entries are specified only on a subset of positions and the goal is to fill in the missing entries so that the resulting matrix is EDM or PSD. If we consider the position of a specified entry as an edge of a graph, then we can associate a graph G to the projection. It appears that the singularity degree of EDM and PSD problems are closely related to the graph G .

The relation between the singularity degree of the completion problem and the underlying graph G has been an active research area. In [9], it is shown that the singularity degree of EDM and PSD completion problem is at most 1 when the underlying graph G is chordal. The EDM completion problem is equivalent to finding frameworks with the specified entries (distances) in Euclidean spaces while the PSD completion is equivalent to finding spherical frameworks on a unit sphere. In [11], it is shown that a generic universal rigid framework in an Euclidean space has singularity degree at most 1. In [22], the worse case singularity degree of a spherical framework underlying a graph G is studied, and it is shown to be at most 1 if the G is chordal. Also in [22], spherical frameworks whose singularity degree grows linearly in the number of vertices are constructed.

3 Singularity degree of faces for non-facially exposed cones

To formally define singularity degree of a face, we first start with an example. We consider a cone whose cross section is a half circle union a square as shown in set C , $0 \notin \text{conv}(C)$, Figure 3.1. The cone $\mathcal{K} = \text{cone}(C) \subset \mathbb{R}^3$, i.e., the closed convex cone generated by the compact *base*, C .

The ray (line) through the point A , $f_A := \text{cone } A \trianglelefteq \mathcal{K}$, represents a face of the convex cone $\mathcal{K} = \text{cone } C$. Any exposed face of \mathcal{K} containing A must also contain the line segment AB . We can find an exposing vector $d_1 \in \mathcal{K}^*$ such that $\text{cone}(AB) = \text{cone}(C) \cap d_1^\perp$, and another exposing vector $d_2 \in (C \cap d_1^\perp)^*$ such that we get

$$\text{cone}(A) = \mathcal{K} \cap d_1^\perp \cap d_2^\perp.$$

Hence we informally define the “singularity degree” of A to be 2, see Definition 3.3 and Algorithm 2, below.

The following lemma plays a key role in defining the singularity degree of a face.

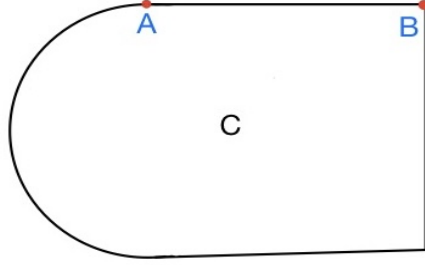


Figure 3.1: cross section of a non facially exposed cone

Algorithm 2 singularity degree of face \mathcal{F}_t of cone \mathcal{K}

Input: A cone $\mathcal{K} \subseteq \mathbb{F}$, a finite dimensional Euclidean space, a proper face $\mathcal{F}_t \trianglelefteq \mathcal{K}$.

Output: Singularity degree d

Set $i \leftarrow 0$, let $\mathcal{F}_i \leftarrow \mathcal{K}$,

Step 1: Find an exposing vector $d_i \in \mathcal{F}_i^*$, $d_i \notin \mathcal{F}_i^\perp$ and $d_i \perp \mathcal{F}_t$

If no such d_i exists, then exit and return $d = i$.

Set $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap d_i^\perp$

$i = i + 1$

Return to Step 1.

Output i .

Lemma 3.1 ([5, Proposition 3.6]). *Let \mathcal{K} be a convex cone and $\mathcal{F} \trianglelefteq \mathcal{K}$, be a proper face. Then $\mathcal{F} \trianglelefteq \mathcal{L} \trianglelefteq \mathcal{K}$ for some exposed face $\mathcal{L} \trianglelefteq \mathcal{K}$, $\mathcal{L} \neq \mathcal{K}$.*

Proof. We include a proof for completeness. Assume \mathcal{F} is proper. Let $x \in \text{relint } \mathcal{F}$. Then, by definition of a face, we get $x \in \partial\mathcal{K}$, the boundary of \mathcal{K} .

By the hyperplane support version of the Hahn-Banach separation theorem there is a hyperplane supporting \mathcal{K} at x , [12]. Therefore there exists $s \in \mathcal{K}^*$ such that

$$\langle s, x \rangle = t \text{ and } \langle s, y \rangle \geq t, \forall y \in \mathcal{K}.$$

We claim that $t = 0$. If $t > 0$, then $\lim_{n \rightarrow \infty} \langle s, \frac{x}{n} \rangle = 0 < t$, a contradiction. Also if $t < 0$, then for n large enough, we have $\langle s, nx \rangle < t$, again a contradiction.

We now claim that $\langle s, \bar{x} \rangle = 0$ for any $\bar{x} \in \mathcal{F}$. Let $\bar{x} \in \mathcal{F}$. Since $x \in \text{relint } \mathcal{F}$, we have $x = \lambda_1 \bar{x} + (1 - \lambda_1)x_1$ for some $x_1 \in \mathcal{F}$ and $\lambda_1 \in (0, 1)$. Therefore $\langle s, x \rangle = 0$ implies $\langle s, \bar{x} \rangle = \langle s, x_1 \rangle = 0$.

From this it follows that $\emptyset \neq \mathcal{F}_1 = s^\perp \cap \mathcal{K} \subset \partial\mathcal{K}$ and \mathcal{F}_1 is an exposed face of \mathcal{K} containing \mathcal{F} . □

Theorem 3.2. *Algorithm 2 terminates in a finite number of steps with $\mathcal{F}_d = \mathcal{F}_t$*

Proof. Since \mathcal{K} is a cone in a finite dimensional Euclidean space, \mathcal{K} is finite dimensional. Since $d_i \notin \mathcal{F}_i^\perp$ at each iteration of Algorithm 2, we have \mathcal{F}_{i+1} is a proper face of \mathcal{F}_i , therefore \mathcal{F}_{i+1} is contained in a proper exposed face of \mathcal{F}_i by Lemma 3.1, hence the dimension is dropped at least by 1. It is easy to see that $\mathcal{F}_t \subseteq \mathcal{F}_i$ at each step. If \mathcal{F}_t is a proper face of \mathcal{F}_d , then there always exists an exposing vector $d_i \in \mathcal{F}_d^*$, $d_i \notin \mathcal{F}_d^\perp$ and $d_i \perp \mathcal{F}_t$, a contradiction. Therefore $\mathcal{F}_d = \mathcal{F}_t$. □

Definition 3.3. *The minimal number d of steps taken in Algorithm 2 to obtain \mathcal{F}_t is defined as the singularity degree of \mathcal{F}_t over \mathcal{K} .*

The minimum number of steps taken in Algorithm 2 can be achieved by a "greedy" strategy, i.e., we seek $d_i \in \text{ri}(\mathcal{F}_i^* \cap \{\mathcal{F}_t\})^\perp$. The following lemma shows the correctness of this strategy. We note a similar result about the facial reduction result is in [14, Proposition 13].

Lemma 3.4. *Assume $\bar{\mathcal{F}} \subseteq \mathcal{F}$, and let $\bar{d} \in \text{ri}(\bar{F}^* \cap \{\mathcal{F}_t\}^\perp)$ and $d \in F^* \cap \{\mathcal{F}_t\}^\perp$. Then $\bar{\mathcal{F}} \cap \bar{d}^\perp \subseteq \mathcal{F} \cap d^\perp$.*

Proof. Since $\bar{\mathcal{F}} \subseteq \mathcal{F}$, we have $\mathcal{F}^* \subseteq \bar{\mathcal{F}}^*$. Hence $d \in \bar{\mathcal{F}}^*$ and $d \in \bar{F}^* \cap \{\mathcal{F}_t\}^\perp$. Since $\bar{d} \in \text{ri}(\bar{F}^* \cap \{\mathcal{F}_t\}^\perp)$, there exists some $\alpha > 1$ such that

$$\alpha \bar{d} + (1 - \alpha)d \in \bar{F}^* \cap \{\mathcal{F}_t\}^\perp$$

Let $x \in \bar{\mathcal{F}} \cap \bar{d}^\perp$, then $\langle x, \alpha \bar{d} + (1 - \alpha)d \rangle \geq 0$. Since $\langle x, \bar{d} \rangle = 0$, we have $\langle x, (1 - \alpha)d \rangle \geq 0$, so $\langle x, d \rangle \leq 0$. Since $d \in \bar{\mathcal{F}}^*$ and $x \in \bar{\mathcal{F}}$, we have $\langle x, d \rangle \geq 0$. Therefore $\langle x, d \rangle = 0$. So $x \in d^\perp$ and we have $\bar{\mathcal{F}} \cap \bar{d}^\perp \subseteq \mathcal{F} \cap d^\perp$. \square

We now have the following proposition following Lemma 3.4.

Proposition 3.5. *The minimum number of steps taken in Algorithm 2 is obtained by computing $d_i \in \text{ri}(\mathcal{F}_i^* \cap \{\mathcal{F}_t\}^\perp)$ in each iteration.*

Proof. Direct consequence from Lemma 3.4. \square

3.1 Connection to linear image of a convex cone

We now connect the singularity degree of a conic optimization problem with the singularity degree of the minimal face in the projected cone. We follow the definition of $\mathbb{E}, \mathbb{F}, \mathcal{M}$ in section 2.1 We start with the following Lemma

Lemma 3.6. *With \mathcal{K}, \mathcal{M} as above, we have:*

1. $\mathcal{M}(\mathcal{K} \cap (\mathcal{M}^*y)^\perp) = \mathcal{M}(\mathcal{K}) \cap \{y\}^\perp$
2. $y_2 \in (\mathcal{M}(\mathcal{K}) \cap y_1^\perp)^* \iff \mathcal{M}^*y_2 \in (\mathcal{K} \cap (\mathcal{M}^*y_1)^\perp)^*$
3. \mathcal{M}^*y exposes a proper face of \mathcal{K} if and only if y exposes a proper face of $\mathcal{M}(\mathcal{K})$.

Proof. 1. Let $z \in \mathcal{M}(\mathcal{F} \cap (\mathcal{M}^*y)^\perp)$, then $z = \mathcal{M}(x)$ for some $x \in \mathcal{F}$ and $\langle x, \mathcal{M}^*y \rangle = \langle \mathcal{M}(x), y \rangle = 0$. Therefore $\mathcal{M}(x) \in y^\perp$. Hence $z \in \mathcal{M}(\mathcal{F}) \cap y^\perp$ and $\mathcal{M}(\mathcal{F} \cap (\mathcal{M}^*y)^\perp) \subseteq \mathcal{M}(\mathcal{F}) \cap \{y\}^\perp$.

For the other direction, let $z \in \mathcal{M}(\mathcal{F}) \cap \{y\}^\perp$. Then $z = \mathcal{M}(x)$ for some $x \in \mathcal{F}$ and $\langle \mathcal{M}(x), y \rangle = \langle x, \mathcal{M}^*y \rangle = 0$. Therefore $x \in (\mathcal{M}^*y)^\perp$. Hence $x \in \mathcal{F} \cap (\mathcal{M}^*y)^\perp$ and $z \in \mathcal{M}(\mathcal{F} \cap (\mathcal{M}^*y)^\perp)$. Therefore $\mathcal{M}(\mathcal{F}) \cap \{y\}^\perp \subseteq \mathcal{M}(\mathcal{F} \cap (\mathcal{M}^*y)^\perp)$ and the proof is complete.

2. $y_2 \in (\mathcal{M}(\mathcal{F}) \cap y_1^\perp)^*$ is same as $y_2 \in (\mathcal{M}(\mathcal{F} \cap (\mathcal{M}^*y_1)^\perp))^*$ by the first equality which is equivalent to $\mathcal{M}^*y_2 \in (\mathcal{F} \cap (\mathcal{M}^*y_1)^\perp)^*$.
3. Assume \mathcal{M}^*y exposes a proper face E of \mathcal{K} , then $E = \mathcal{K} \cap (\mathcal{M}^*y)^\perp$ where $\mathcal{M}^*y \in \mathcal{K}^*$ and $\mathcal{M}^*y \notin \mathcal{K}^\perp$. So $y \in \mathcal{M}(\mathcal{K})^*$ and there exists some $x \in \mathcal{K}$ such that $\langle \mathcal{M}^*y, x \rangle = \langle y, \mathcal{M}(x) \rangle \neq 0$. Hence $y \notin \mathcal{M}(\mathcal{K})^\perp$. Therefore $F = \mathcal{M}(E) = \mathcal{M}(\mathcal{K}) \cap y^\perp$ is a proper face of $\mathcal{M}(\mathcal{K})$.

For the other direction, assume y exposes a proper face of $\mathcal{M}(\mathcal{K})$, then $F = \mathcal{M}(\mathcal{K}) \cap y^\perp$ where $y \in \mathcal{M}(\mathcal{K})^*$ and $y \notin \mathcal{M}(\mathcal{K})^\perp$. Hence $\mathcal{M}^*y \in \mathcal{K}^*$ and there exists some $x \in \mathcal{K}$ such that $\langle y, \mathcal{M}(x) \rangle = \langle \mathcal{M}^*y, x \rangle \notin 0$. So $\mathcal{M}^*y \notin \mathcal{K}^\perp$. Also $F = \mathcal{M}(\mathcal{K}) \cap y^\perp = \mathcal{M}(\mathcal{K} \cap (\mathcal{M}^*y)^\perp)$. F is not empty implies $\mathcal{K} \cap (\mathcal{M}^*y)^\perp$ is not empty. So \mathcal{M}^*y exposes a proper face of \mathcal{K} . \square

Theorem 3.7. *Let $\mathcal{M} : \mathbb{E} \rightarrow \mathbb{F}$ be a linear transformation between two finite dimensional Euclidean spaces \mathbb{E} and \mathbb{F} . Let $\mathcal{K} \subset \mathbb{E}$ be a convex cone, $b \in \mathbb{F}$ and*

$$\mathcal{F} := \{X \in \mathcal{K} : \mathcal{M}(X) = b\}.$$

Let $v_1, v_2, \dots, v_r \in b^\perp$, and $v_1 \in \mathcal{M}(\mathcal{K})^*$, $v_2 \in (\mathcal{M}(\mathcal{K}) \cap v_1^\perp)^*$, \dots , $v_r \in (\mathcal{M}(\mathcal{K}) \cap \dots \cap v_{r-1}^\perp)^*$ and

$$N := \mathcal{M}(\mathcal{K}) \cap v_1^\perp \cap \dots \cap v_r^\perp, \quad E := \mathcal{K} \cap \mathcal{M}^*(v_1)^\perp \cap \dots \cap \mathcal{M}^*(v_r)^\perp.$$

Then:

1. v_i exposes a face of $\mathcal{M}(\mathcal{K}) \cap v_1^\perp \cap \dots \cap v_{i-1}^\perp$ containing b and $\mathcal{M}^*(v_i)$ exposes a face of $\mathcal{K} \cap \mathcal{M}^*(v_1)^\perp \cap \dots \cap \mathcal{M}^*(v_{i-1})^\perp$ containing F .
2. N is a face of $\mathcal{M}(\mathcal{K})$ containing b and E is a face of \mathcal{K} containing F ;
3. $N = \text{face}(b, \mathcal{M}(\mathcal{K}))$ if, and only if, $E = \text{face}(F, \mathcal{K})$.

Proof. 1. Since $v_1 \in \mathcal{M}(\mathcal{K})^*$, $v_2 \in (\mathcal{M}(\mathcal{K}) \cap v_1^\perp)^*$, \dots , $v_r \in (\mathcal{M}(\mathcal{K}) \cap \dots \cap v_{r-1}^\perp)^*$, then by Lemma 3.6 we have $\mathcal{M}^*v_1 \in \mathcal{K}^*$, $\mathcal{M}^*v_2 \in (\mathcal{K} \cap (\mathcal{M}^*v_1)^\perp)^*$, \dots , $\mathcal{M}^*v_r \in (\mathcal{K} \cap \dots \cap (\mathcal{M}^*v_{r-1})^\perp)^*$. Also $\langle v_i, b \rangle = \langle v_i, \mathcal{M}(X) \rangle = \langle \mathcal{M}^*v_i, X \rangle = 0$ for any $X \in F$. So v_1 exposes a face of $\mathcal{M}(\mathcal{K})$ containing b , \mathcal{M}^*v_1 exposes a face of \mathcal{K} containing F , v_2 exposes a face of $\mathcal{M}(\mathcal{K}) \cap v_1^\perp$ containing b , \mathcal{M}^*v_2 exposes a face of $\mathcal{K} \cap (\mathcal{M}^*v_1)^\perp$ containing F ,... The conclusion follows through.

2. The conclusion follows from the property that a face of a face is also a face of the original cone.
3. Suppose $N = \text{face}(b, \mathcal{M}(\mathcal{K}))$ and to the contrary E is not the minimal face of \mathcal{K} containing F . Then by the theorem of alternative, there exists some nonzero v such that $\mathcal{M}^*v \in E^*$ exposing a proper face of $E = \mathcal{K} \cap \mathcal{M}^*(v_1)^\perp \cap \dots \cap \mathcal{M}^*(v_r)^\perp$ containing F . By Lemma 3.6, v exposes a proper face of $N = \mathcal{M}(\mathcal{K}) \cap v_1^\perp \cap \dots \cap v_r^\perp$ containing b . However, this is impossible since N is already the minimal face of $\mathcal{M}(\mathcal{K})$ containing b . For the other direction, assume E is the minimal face of \mathcal{K} containing F , suppose to the contrary that N is not the minimal face of $\mathcal{M}(\mathcal{K})$ containing b , then by Lemma 3.1, there exists $v \in N^*$ exposing a proper face of E containing b . Therefore \mathcal{M}^*v exposes a proper face of E containing F , a contradiction. \square

Corollary 3.8. *The singularity degree of the system $\mathcal{F} := \{X \in \mathcal{K} : \mathcal{M}(X) = b\}$ in \mathbb{E} is equal to the singularity degree of $N = \text{face}(b, \mathcal{M}(\mathcal{K}))$ in \mathbb{F} .*

Proof. This is a direct consequence from Lemma 3.6 and Theorem 3.7. \square

3.2 The nullspace form

We consider $\mathcal{F} := \{X \in \mathcal{K} : \mathcal{M}(X) = b\}$ in its nullspace form, i.e., let $\mathcal{F} := \{Z \in \mathcal{K} : \mathcal{A}^*y + Z - C = 0, y \in \mathcal{F}\}$ where \mathcal{F} is a finite dimensional linear space and \mathcal{A} is a linear transformation from E to \mathcal{F} . We define $L = \mathcal{A}^*(\mathcal{F})$ and let \mathcal{K}/L be the quotient cone and C/L be a point on the quotient cone \mathcal{K}/L . Then we have the following corollary.

Corollary 3.9. *The singularity degree of the system $\mathcal{F} := \{Z \in \mathcal{K} : \mathcal{A}^*y + Z - C = 0, y \in \mathcal{F}\}$ is equal to the singularity degree of $\text{face}(C/L, \mathcal{K}/L)$, which is equal to the singularity degree of the nullspace form of the system \mathcal{F} .*

Proof. The first part of this theorem is just a reformulation of Theorem 3.7 in the nullspace form, and the proof is similar to the proof of Theorem 3.7.

For the second part, we just need to show there is an isomorphism between the two convex cones $\mathcal{M}(\mathcal{K})$ and \mathcal{K}/L . For any X such that $\mathcal{M}(X) = b$, X can be written as $X = \mathcal{A}^*y + C$ where $\mathcal{M}(\mathcal{A}^*y) = 0$ and $\mathcal{M}(C) = b$. The isomorphism is then given by $\mathcal{M} : \mathcal{K}/L \rightarrow \mathcal{M}(\mathcal{K})$

Let \bar{p}, \bar{q} be two distinct points on \mathcal{K}/L such that $\bar{p} = p + \mathcal{A}^*y_1$ and $\bar{q} = q + \mathcal{A}^*y_2$ such that $\mathcal{M}(p - q) \neq 0$. Then $\mathcal{M}(\bar{p} - \bar{q}) = \mathcal{M}(p - q) \neq 0$. So the homomorphism \mathcal{M} from \mathcal{K}/L to \mathcal{K} is one-to-one. It is easy to see it is also an onto map. Therefore it is an isomorphism. \square

Remark 3.10. *By combining Lemma 3.4, Proposition 3.5 and Corollary 3.8, we can show that the minimum number of facial reduction in Algorithm 2 is achieved by taking $d_i = \mathcal{M}^*v_i \in \text{ri}(\mathcal{F}_i^* \cap \{b\}^\perp)$ at each step, a result which is proven in its nullspace form in [14].*

4 Bounding singularity degree for frameworks

We begin with the definition of frameworks.

Definition 4.1. *A configuration $p : V \rightarrow \mathbb{R}^d$ of a graph $G = (V, E)$ assigns positions in d -dimensional Euclidean space to the vertices of G . A framework is an ordered pair (G, p) consisting of a graph G and a configuration p .*

Given a framework, one can define its stress and stress matrices.

Definition 4.2. *A stress is an assignment $\omega : E \rightarrow \mathbb{R}$ of weights to the edges of (G, p) . An equilibrium stress is a stress such that*

$$\sum_{j: v_i v_j \in E} \omega_{ij} (p(v_i) - p(v_j)) = 0.$$

for all $v_i \in V$; i.e., the net stress at each vertex is 0.

Definition 4.3. *Given a stress ω for (G, p) in \mathbb{R}^d , we define $\Omega = \Omega(\omega)$ to be the $|V| \times |V|$ symmetric matrix with off-diagonal entries $-\omega_{ij}$ and diagonal entries $\sum_j \omega_{ij}$.*

In [7], the connection between facial reduction and level of PSD stress matrices is established. The sequence of stress matrices computed at each step of facial reduction algorithm provide a certificate of dimensional rigidity and universal rigidity. The minimum number of the level of stress matrices is indeed the singularity degree of the corresponding SDP optimization problem. See also the book [2]. We call the *singularity degree of a framework* to be the minimal number of level stress matrices it has

So far, little has been known about the singularity degree of frameworks underlying a particular graph G . In [7] the stress matrices and its connection to facial reduction is studied in

the nullspace form . In [9], it is shown that the singularity degree of the matrix completion problem for chordal graphs is at most 1. We show that stress matrices are exactly the solutions to the auxiliary problem of the matrix completion problem in its nullspace form. Therefore, a framework underlying a chordal graph can have at most 1 level of stress matrices.

Given a framework (G, p) where $G = (V, E)$, $|V| = n$, let $d_{ij} = \|p_i - p_j\|^2, p_i \in \mathbb{R}^r, \forall i, j \in E$. We let $d_E = \Pi_E(d) \in \mathbb{R}^E$. Let \hat{X} be a (centered) feasible solution for the following EDM completion problem:

$$\text{find } \hat{X} \in \mathcal{S}_+^n \text{ such that } \Pi_E(K(\hat{X})) = d_E, \hat{X}e = 0. \quad (4.1)$$

Here $K(\hat{X})$ is an EDM, a solution of the EDM completion problem, where $K(\hat{X})_{ij} = X_{ii} + X_{jj} - 2X_{ij}$. If in addition \hat{X} is a unique feasible solution, then the framework (G, p) is called *universally rigid*. Necessarily, $\text{rank}(\hat{X}) \leq r$. We note that $\hat{X}e = 0$ and $\hat{X} = PP^T = PQQ^T P^T$, for any Q orthogonal. Thus the points are centered, $P^T e = 0$, but we can allow rotations using Q .

The auxiliary problem from theorem of the alternative, that identifies positive singularity degree for (4.1), is the following:

$$0 \neq Z = K^*(\Pi_E^*(v)) \succeq 0, \langle v, d \rangle = 0.$$

The singularity degree of problem (4.1) is called the *singularity degree* of EDM completion.

In fact we can reformulate the EDM completion problem (4.1) in the nullspace formulation. We use the feasible $\hat{X} \in \mathcal{S}_+^n$ given above. Now the problem is equivalent to finding y such that

$$\hat{X} + \mathcal{N}(y) = J\hat{X}J + \sum_{ij \in E^c} y_{ij} J E_{ij} J \succeq 0, \quad (4.2)$$

where we note that $\text{range}(\mathcal{N}) = \text{null}(\Pi_E \circ K) \cap \mathcal{S}_c^n$.

The auxiliary problem (theorem of the alternative) for (4.2) is

$$\begin{aligned} \langle \bar{\Omega}, J\hat{X}J \rangle &= 0 \\ \langle J E_{ij} J, \bar{\Omega} \rangle &= 0, \forall ij \in E^c \\ \bar{\Omega} &\in \mathcal{S}_+^n \end{aligned}$$

The centered solution $\Omega = J\bar{\Omega}J$ can be shown to be a PSD equilibrium stress matrix. In the following, we will just refer PSD equilibrium stress matrix as *PSD stress matrix*. Since both \hat{X} and Ω is centered, we have

$$\text{rank}(\hat{X}) + \text{rank}(\Omega) \leq n - 1. \quad (4.3)$$

By doing facial reductions on problem 4.2, we can obtain a sequence of PSD stress matrices $\Omega_1, \Omega_2, \dots, \Omega_k$ by using the previous PSD stress matrix to reduce the problem to an equivalent problem over a smaller SDP cone, and the following relation holds.

$$\text{rank}(\hat{X}) + \text{rank}(\Omega_1) + \dots + \text{rank}(\Omega_k) \leq n - 1. \quad (4.4)$$

Lemma 4.4. *Given a framework (G, p) , The minimal number of (nonzero) PSD stress matrices of (G, p) is equal to the singularity degree of the corresponding EDM completion problem.*

Proof. The singularity degree of framework (G, p) is the singularity degree of problem 4.2, which is just the nullspace form of the EDM completion problem 4.1. Therefore the two singularity degrees are equal. \square

We can also characterize universal rigidity using sequence of PSD stress matrices. Given a framework (G, p) whose affine span has dimension d . Then (G, p) is called *dimensionally rigid* if there exists a sequence of PSD stress matrices $\Omega_1, \Omega_2, \dots, \Omega_k$ such that

$$d + \text{rank}(\Omega_1) + \dots + \text{rank}(\Omega_k) = n - 1. \quad (4.5)$$

If in addition the member directions $\{p_i - p_j\}_{\{i,j\} \in E_G}$ do not lie in a conic (there exists no non-trivial affine flex for (G, p)), then it is universally rigid [7].

The singularity degree of the matrix completion problem is closely related to the stability of frameworks. A well-known bound for the forward error (distance to the original solution set) in terms of the backward error (amount of constraints violation) of LMI is given in [21] in the form

$$\text{forward error} = O((\text{backward error})^{2-s}) \quad (4.6)$$

where s is the singularity degree of the LMI.

Given a framework (G, p) whose affine span has dimension d , if there exists at most one level of stress matrix Ω such that $\text{rank}(\Omega) + d = n - 1$, and the member directions $\{p_i - p_j\}_{\{i,j\} \in E_G}$ do not lie in a conic (there exists no non-trivial affine flex for (G, p)), the framework is called *super stable* [6–8]. Hence, super stable framework is just universally rigid frameworks with at most one level of PSD stress matrix. It is conjectured in [1] and proved in [11] that generic universal rigid frameworks has singularity at most 1.

4.1 Frameworks underlying chordal graphs

Chordal graphs appear very often in the study of EDM completion problems. A graph is called chordal if any cycle of four or more nodes (vertices) has a chord, i.e., an edge exists joining any two nodes that are not adjacent in the cycle.

The following fundamental theorem was proved by Bakonyi and Johnson [3].

Theorem 4.5. *Let G be a chordal graph, A matrix $A \in S_G^n$ has an EDM completion if and only if for every clique χ of the graph, the matrix A_χ is a Euclidean distance matrix.*

The following theorem about the singularity of EDM completions of chordal graphs was proved in [9].

Theorem 4.6. *For chordal graphs, the singularity degree of the EDM completion problem is at most 1.*

We now have the following conclusion about the level of stress matrices of a framework underlying chordal graphs.

Corollary 4.7. *Given a framework (G, p) . If G is chordal, then the level of PSD stress matrices is at most 1.*

Proof. Consequence of Lemma 4.4 and Theorem 4.5. □

4.2 Frameworks underlying Laman graphs

Laman graphs have played in the study of rigidity theory. In particular, by restricting ourselves within generic configurations, a framework is minimally rigid if and only if the underlying graph is Laman [8, 13].

We start with the formal definition of Laman graphs.

Definition 4.8 (Laman graphs). A graph $G = (V, E)$ is Laman if $|E| = 2|V| - 3$ and every subgraph of G with k vertices has at most $2k - 3$ edges.

Definition 4.9. Let p be a configuration of n points in \mathbb{R}^d . We say that p is generic if the coordinates of p do not satisfy any polynomial with rational coefficients.

Theorem 4.10. Let $G = (V, E)$ be a Laman-sparse graph with n vertices, let $p = \{p_1, \dots, p_n\}$ be a 2d configuration of G , let $d_{ij} = \|p_i - p_j\|^2, \forall ij \in E$ be the squared distance between p_i and p_j and d_E be a point in \mathbb{R}^E whose coordinates are squared distances in the edge set E . If p is generic, then d_E is also generic.

Proof. Suppose to the contrary that d_E is not generic, then there exists a nonzero polynomial f such that $f(d_1, \dots, d_{|E|}) = 0$. Hence $f(\dots, \|p_i - p_j\|^2, \dots) = 0$, which we can rewrite it as $g(p_1, \dots, p_n) = 0 = f(\dots, \|p_i - p_j\|^2, \dots)$. Since p is generic, we know $g \equiv 0$. (g is a zero polynomial)

Since $g \equiv 0$, we know $f(d_1, \dots, d_{|E|}) = 0$ for any point $(d_1, \dots, d_{|E|})$ where $(d_1, \dots, d_{|E|})$ are squared distances of some 2D framework (G, p) . Let D_E be the set of all those points $(d_1, \dots, d_{|E|})$. Thus $f \equiv 0$ in the Zariski closure of D_E (over the complex). Since G is Laman-sparse, E is independent in the algebraic matroid of the 2D Cayley-Menger ideal $\text{CM}(n, 2)$ [15]. Thus the Zariski closure of D_E has dimension $|E|$. Since f has $|E|$ variables, $f \equiv 0$ on a complex variety of dimension $|E|$ implies f is a zero polynomial, a contradiction. Hence d_E has to be generic. \square

Theorem 4.11. Let (G, p) be a generic framework of n vertices in \mathbb{R}^2 . If G is Laman-sparse, then the level of stress matrices for (G, p) is zero. In other words, if we are given the set of squared distance d of (G, p) corresponding to the edge set E where p is generic, then we can find a framework \bar{p} of maximal dimension $(n-1)$ such that $\|\bar{p}_i - \bar{p}_j\|^2 = d_{ij}, \forall ij \in E$.

Proof. Since (G, p) is a generic framework in \mathbb{R}^2 , let X_p be the corresponding gram matrix of p , by Theorem 4.10, we know $d_E = \Pi_E(K(X_p))$ is also generic. Since d_E is generic, we know the singularity degree of $\text{face}(d_E, \Pi_E(K(\mathcal{S}_+^n)))$ is zero. Otherwise, suppose the singularity degree of $\text{face}(d_E, \Pi_E(K(\mathcal{S}_+^n)))$ is 1 or more, then d_E must lie on the boundary of $\Pi_E(K(\mathcal{S}_+^n))$. However, since \mathcal{S}_+^n is a semialgebraic set, $\Pi_E(K(\mathcal{S}_+^n))$ is also semialgebraic, any point on the boundary of $\Pi_E(K(\mathcal{S}_+^n))$ will be in the zero set of some polynomial $p(X)$ over $Q[X]$, therefore it must be non-generic, a contradiction. Therefore the singularity degree of $\text{face}(d_E, \Pi_E(K(\mathcal{S}_+^n)))$ is zero. Thus by Corollary 3.8 and Corollary 3.9, we know (G, p) does not have a nonzero PSD stress matrix, hence the corresponding EDM completion problem admits a maximal rank solution \bar{p} . \square

We now study the singularity degree of generic framework (G, p) where G is a Laman graph plus one edge. To begin with, we introduce the following lemma.

Lemma 4.12. Let f be a polynomial in $Q[x_1, \dots, x_n]$. If there exist x, y with $x \neq y$ such that

$$f(x) = f(y) = 0 \implies f(\lambda x + (1 - \lambda)y) = 0, \quad \forall \lambda \in [0, 1]$$

Then

$$f(\lambda x + (1 - \lambda)y) = 0, \quad \forall \lambda \in \mathbb{R}$$

In other words, if the zero set of a polynomial contains a face, then it must contain the whole affine span of the face.

Proof. Without loss, let $z = tx + (1 - t)y$ where $t > 1$. Let $d = x - y$, and $z_0 = \frac{1}{2}x + \frac{1}{2}y$. So $f(z_0) = 0$ Then we can compute $f(z)$ by the Taylor expansion at z_0 such that $f(z) = f(z_0) + \nabla_d f(z_0)(z - z_0) + \nabla_d^2 f(z_0)(z - z_0)^2/2 + \dots + \nabla_d^{(m)} f(\bar{z})(z - z_0)^m/m!$. It is easy to see $\nabla_d^{(i)} f(z_0) = 0$ for any i . Also since f is a polynomial, we have $\nabla_d^{(m)} f(\bar{z}) = 0$ for m sufficiently large. Hence $f(z) = 0$, the lemma is proved. \square

Proposition 4.13. *A generic 2D framework (G, p) where the underlying graph G is a Laman graph plus one edge have singularity degree exactly equal to 1.*

Proof. Let $d_{E_1} = (d_1, \dots, d_{|E_1|})$ be the point whose components are squared distances of a generic framework (G, p) where the edge set of G is $E_1 = E \cup e$. Assume G is a Laman plus one graph and E is the set of the Laman graph while e is the extra one edge. We define $\phi : p \rightarrow d_{E_1}$ to be the map from frameworks to squared distances where $\phi_{ij} = \|p_i - p_j\|^2$. Let $\bar{d}_{E_1} = \phi(\bar{p})$.

Since (G, \bar{p}) is a generic framework, the Jacobian matrix of the map ϕ has maximum rank at \bar{p} which is $2n - 3$. By the constant rank theorem, locally there is an open set U containing \bar{p} such that $\phi(U)$ is a submanifold containing \bar{d}_{E_1} of dimension $|E| = 2n - 3$. Hence \bar{d}_{E_1} is on the boundary of $\Pi_{E_1}(K(\mathcal{S}_+^n))$, so the singularity degree is at least 1.

By Tarski-Seidenberg theorem, $\Pi_{E_1}(K(\mathcal{S}_+^n))$ is a semialgebraic set generated by equalities and inequalities in $Q[X]$. Now suppose the minimal face containing \bar{d}_{E_1} in $\Pi_{E_1}(K(\mathcal{S}_+^n))$ has singularity degree 2 or more. Without loss assume $\bar{d}_{E_1} \in \mathcal{F}_2 \subsetneq \mathcal{F}_1 \subsetneq \Pi_{E_1}(K(\mathcal{S}_+^n))$ where \mathcal{F}_1 is a proper face of \mathcal{F}_2 and \mathcal{F}_2 is a proper face of $\Pi_{E_1}(K(\mathcal{S}_+^n))$. Then obviously \mathcal{F}_1 lies on the boundary of $\Pi_{E_1}(K(\mathcal{S}_+^n))$, so there exists some polynomial $f_1(X)$ in $Q[X]$ such that $f_1(\mathcal{F}_1) = 0$. Pick a point d^* in the relative interior of \mathcal{F}_1 , then the line L connecting d^* and \bar{d}_{E_1} is contained in the zero set of f_1 by Lemma 4.12. Now since \mathcal{F}_2 is on the boundary of \mathcal{F}_1 , \bar{d}_{E_1} is also on the boundary of \mathcal{F}_1 , we take all the polynomials defining the boundary of the semialgebraic set $\Pi_{E_1}(K(\mathcal{S}_+^n))$ that vanish on \mathcal{F}_2 , among them there must exist another polynomial $f_2(X)$ in $Q[X]$ such that the zero set of f_2 intersects L properly at \bar{d}_{E_1} (otherwise the zero set of f_2 contains the entire line L , therefore \bar{d}_{E_1} can not be on the boundary of \mathcal{F}_1). Since the zero set of f_1 contains the line L while the zero set of f_2 intersects L properly, the local dimension of the intersection of the zero sets of f_1 and f_2 at \bar{d}_{E_1} is at less one less than the local dimension of the zero sets of f_1 at \bar{d}_{E_1} . Hence the dimension of the intersection of the zero sets of f_1 and f_2 at \bar{d}_{E_1} is at most $2n - 4$ locally.

Now since $f_1(\bar{d}_{E_1}) = 0$, we have $f_1(\dots, \|\bar{p}_i - \bar{p}_j\|^2, \dots) = g_1(\bar{p}_1, \dots, \bar{p}_n) = 0$. Since \bar{p} is algebraic independent over Q , g_1 must be a zero polynomial. Therefore pick any $d^* \in \phi(U)$, we have $f_1(d^*) = f_1(\dots, \|p_i^* - p_j^*\|^2, \dots) = g_1(p_1^*, \dots, p_n^*) = 0$. Similarly $f_2(d^*) = f_2(\dots, \|p_i^* - p_j^*\|^2, \dots) = g_2(p_1^*, \dots, p_n^*) = 0$. However $\phi(U)$ has dimension $2n - 3$, a contradiction to the previous conclusion that the local dimension of the intersection of f_1 and f_2 at \bar{d}_{E_1} is at most $2n - 4$ locally. \square

Remark 4.14. *In [4], it is shown that if G is a Laman circuit, then there are generic 2-dimensional frameworks (G, p) satisfying a non-zero PSD stress. So we have shown that in this case, there is exactly one level of non-zero PSD stress for a generic 2-dimensional framework (G, p) .*

Proposition 4.15. *A generic 2D framework (G, p) where the underlying graph G is a Laman graph plus d edges have singularity at most d .*

Proof. The proof is similar to the proof of the previous theorem. Let $\bar{d}_{E_1} = (d_1, \dots, d_{|E_1|})$ be the point whose components are squared distances of a generic framework (G, \bar{p}) where the edge set of G is $E_1 = E \cup E_d$. Assume G is a Laman plus d graph and E is the set set of the Laman graph while E_d is the extra d edges. We define $\phi : p \rightarrow d_{E_1}$ to be the map from frameworks to squared distances where $\phi_{ij} = \|p_i - p_j\|^2$. Let $\bar{d}_{E_1} = \phi(\bar{p})$.

Now suppose the minimal face containing \bar{d}_{E_1} in $\Pi_{E_1}(K(\mathcal{S}_+^n))$ has singularity degree $d + 1$ or more. Without loss assume $\bar{d}_{E_1} \in \mathcal{F}_{d+1} \subseteq \dots \subsetneq \mathcal{F}_1 \subsetneq \Pi_{E_1}(K(\mathcal{S}_+^n))$. By a similar argument, we can find polynomials f_1, \dots, f_{d+1} in $Q[X]$ and we claim the intersection of f_1, \dots, f_{d+1} at \bar{d}_{E_1} has dimension at most $2n - 4$. We prove the claim by induction. We assume \mathcal{F}_i is contained in the zero set of f_1, \dots, f_i and the intersection of f_1, \dots, f_i at \bar{d}_{E_1} has dimension at most $2n - 3 + d - i$

locally. Now \mathcal{F}_{i+1} is on the boundary of \mathcal{F}_i . Pick a point d^* in the relative interior of \mathcal{F}_i and let L be the line connecting d^* and \bar{d}_{E_1} . Then there exists a polynomial f_{i+1} which vanishes on \mathcal{F}_{i+1} such that f_{i+1} intersects L properly. Hence the dimension of the intersection of f_1, \dots, f_i with f_{i+1} is at least one less than the intersection of f_1, \dots, f_i at \bar{d}_{E_1} . Therefore the intersection of f_1, \dots, f_i, f_{i+1} at \bar{d}_{E_1} has dimension at most $2n - 3 + d - i - 1$. The induction hypothesis holds. Hence the intersection of f_1, \dots, f_{d+1} at \bar{d}_{E_1} has dimension at most $2n - 3 + d - d - 1 = 2n - 4$.

Again since (G, \bar{p}) is a generic framework, the Jacobian matrix of the map ϕ has maximum rank at \bar{p} which is $2n - 3$. So locally there is an open set U containing \bar{p} such that $\phi(U)$ is a submanifold containing \bar{d}_{E_1} of dimension $|E| = 2n - 3$. Then $f_1(\bar{d}_{E_1}) = f_2(\bar{d}_{E_1}) = \dots = f_{d+1}(\bar{d}_{E_1}) = 0$ and due to algebraic independence of \bar{p} we have $f_1(d^*) = f_2(d^*) = \dots = f_{d+1}(d^*) = 0$ for any $d^* \in \phi(U)$. A contradiction to the previous conclusion that the intersection of f_1, \dots, f_{d+1} at \bar{d}_{E_1} has dimension at most $2n - 3 + d - d - 1 = 2n - 4$ locally. \square

Example 1. Consider the EDM completion problem where G is K_4

$$\begin{aligned} \text{find } X &\succeq 0 \\ \text{s.t. } \Pi_E(K(X)) &= d_E \end{aligned}$$

where $E = \{12, 13, 14, 23, 24, 34\}$, $X \in \mathcal{S}_+^4$ and d_E is the set of distances from a generic configuration $P = [p_1, p_2, p_3, p_4]^T$ as shown in Figure 4.1. The stress matrix has the following form

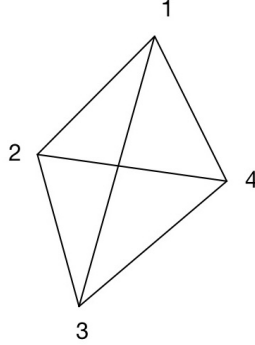


Figure 4.1: framework underlying K_4

$$\mathcal{F} = \begin{pmatrix} w_{12} + w_{13} + w_{14} & -w_{12} & -w_{13} & -w_{14} \\ -w_{12} & w_{12} + w_{23} + w_{24} & -w_{23} & -w_{24} \\ -w_{13} & -w_{23} & w_{13} + w_{23} + w_{34} & -w_{34} \\ -w_{14} & -w_{24} & -w_{34} & w_{14} + w_{24} + w_{34} \end{pmatrix}$$

where $w_{12}, w_{23}, w_{14}, w_{34}$ are outward stresses (positive signs) and w_{24}, w_{13} are negative stresses (negative signs).

It can be seen there must exist at least one nonzero PSD stress matrix as the framework with distance d_E can not be lifted to higher dimension. If vertex 1 is rotated around the edge 24 to lift the framework to dimension 3, the distance between vertex 1 and vertex 3 would shrink. Also since $\text{rank}(PP^T) = 2$ and the rank of a nonzero PSD stress matrix is at least 1, we have $\text{rank}(PP^T) + \text{rank}(\mathcal{F}) = 3 = n - 1$ where $n = 4$ is the number of vertices. By (4.3), $\text{rank}(PP^T) + \text{rank}(\mathcal{F})$ is already maximal, hence there is only 1 level of PSD stress matrix.

5 Bounding singularity degree for tensegrities

The above results can be easily extended to linear conic optimization problems where inequalities are included by adding auxiliary variables.

5.1 Tensegrity

A tensegrity (G, p) is a framework where some edges (bars) are replaced by cables, which are constrained to not get longer in length, and some bars are replaced by struts, which are constrained to not get shorter in length. Given a graph G with edge sets for bars, cables and struts, a tensegrity underlying G is a feasible solution of the following linear SDP feasibility problem.

$$\begin{aligned} & \text{find } \hat{X} \in \mathcal{S}_+^n \\ & \text{such that } \Pi_{E_0}(K(\hat{X})) = d_{E_0}, \\ & \quad \Pi_{E_1}(K(\hat{X})) \leq d_{E_1}, \\ & \quad \Pi_{E_2}(K(\hat{X})) \geq d_{E_2}, \hat{X}e = 0. \end{aligned} \tag{5.1}$$

where $E = E_0 \cup E_1 \cup E_2$, E_0 is the edge set for bars, E_1 is the edge sets for cable, and E_2 is the edge set for struts.

The auxiliary problem for (5.1) is then

$$\begin{aligned} \langle \bar{\Omega}, J\hat{X}J \rangle &= 0 \\ \langle JE_{ij}J, \bar{\Omega} \rangle &= 0, \forall ij \in (E_0 \cup E_1 \cup E_2)^c \\ \langle JE_{ij}J, \bar{\Omega} \rangle &\geq 0, \forall ij \in E_1 \\ \langle JE_{ij}J, \bar{\Omega} \rangle &\leq 0, \forall ij \in E_2 \\ \bar{\Omega} &\in \mathcal{S}_+^n \end{aligned}$$

A PSD stress matrix $\mathcal{F} = J\bar{\mathcal{F}}J$ is called *proper* if it satisfies the above auxiliary problem or equivalently \mathcal{F} is a PSD stress matrix for the framework and $\mathcal{F}_{ij} \geq 0$, when $\{i, j\}$ is a cable, and $\mathcal{F}_{ij} \leq 0$, when $\{i, j\}$ is a strut.

We add auxiliary variables to problem (5.1)

$$\begin{aligned} & \text{find } \hat{X} \in \mathcal{S}_+^n \\ & \text{such that } \Pi_{E_0}(K(\hat{X})) = d_{E_0}, \\ & \quad \Pi_{E_1}(K(\hat{X})) + z_{E_1} = d_{E_1}, \\ & \quad \Pi_{E_2}(K(\hat{X})) - z_{E_2} = d_{E_2}, \hat{X}e = 0. \\ & \quad z_{E_1} \geq 0, z_{E_2} \geq 0 \end{aligned} \tag{5.2}$$

Let $\mathcal{K}_2 = (0^{|E_0|}, \mathbb{R}_{\geq 0}^{|E_1|}, \mathbb{R}_{\leq 0}^{|E_2|})$ and $\mathcal{K}_1 = \Pi_E(K(\mathcal{S}_+^n))$. The singularity degree of problem (5.1) is then the minimal face containing $(d_{E_0}, d_{E_1}, d_{E_2})^T$ over the cone $\mathcal{K}_1 + \mathcal{K}_2$ where the + sign is the Minkowski sum. It is easy to see $\mathcal{K}_1 + \mathcal{K}_2$ is a semialgebraic set since \mathcal{K}_1 is semialgebraic. The boundary of $\mathcal{K}_1 + \mathcal{K}_2$ is generated by polynomials in $Q[X]$.

Proposition 5.1. *A generic tensegrity framework (G, p) where the underlying graph G is a Laman graph plus d edges have singularity at most d .*

Proof. The proof follows by replacing $\Pi_{E_1}(K(\mathcal{S}_+^n))$ with $\mathcal{K}_1 + \mathcal{K}_2$ in the proof of Proposition 4.15. \square

6 Conclusion

We define the singularity degree of any face of a convex cone as the minimum number of steps to expose it by using exposing vectors in a sequence of growing dual cones. Given a convex cone, the linear image of it is also a convex cone. However, the linear image of a facially exposed convex cone is not necessarily facially exposed. We show that the singularity degree of the minimum face containing vector b is exactly the singularity degree of the linear conic optimization problem $\{\min c^T x \mid \mathcal{M}(x) = b, x \in \mathcal{K}\}$. We then utilize tools from algebraic geometry and rigidity theory to give an upper bound of the singularity degree of the EDM completion problem related to frameworks or tensesities in Euclidean space. The most interesting case seems to be the case where G is a Laman plus 2 graph, as this is the first case where strict complementarity could possibly fail for a generic framework. So far, all the frameworks where strict complementarity fails in the literature are constructed in a way that the coordinates of some points are in some special positions, i.e., the framework is non-generic. It is an interesting open problem to construct generic frameworks with singularity degree more than 1. Although our method is focused on EDM completion problem, it can be adjusted to be analyze other similar SDP problems.

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