ACM Communications in Computer Algebra, Vol. 51, No. 1, Issue 199, March 2017

Finding Maximum Rank Moment Matrices by Facial Reduction on Primal Form and Douglas-Rachford Iteration

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Abstract

Recent breakthroughs have been made in the use of semi-definite programming and its application to real polynomial solving. For example, the real radical of a zero dimensional ideal, can be determined by such approaches. Some progress has been made on the determination of the real radical in positive dimension by Ma, Wang and Zhi[5, 4]. Such work involves the determination of maximal rank semidefinite matrices. Existing methods are computationally expensive and have poorer accuracy on larger examples.

In previous work we showed that regularity in the form of the Slater constraint qualification (strict feasibility) fails for the moment matrix in the SDP feasibility problem[6]. We used facial reduction to obtain a smaller regularized problem for which strict feasibility holds. However we did not have a theoretical guarantee that our methods, based on facial reduction and Douglas-Rachford iteration ensured the satisfaction of the maximum rank condition.

Our work is motivated by the problems above. We discuss how to compute the moment matrix and its kernel using facial reduction techniques where the maximum rank property can be guaranteed by solving the dual problem. The facial reduction algorithms on the primal form is presented. We give examples that exhibit for the first time additional facial reductions beyond the first which are effective in practice with much better accuracy than SeDuMi(CVX).

1 Introduction

Our SDP feasibility formulation is a moment problem equivalent to finding M for a linear system of the following type

$$\mathcal{A}(M) = b, \quad B^T M = 0, \quad M \in \mathcal{S}_+^k, \tag{1}$$

where $\mathcal{S}_+^k$ denotes the convex cone of $k \times k$ real symmetric positive semi-definite matrices, B is the coefficient matrix of a polynomial system and $\mathcal{A} : \mathcal{S}_+^k \to \mathcal{R}^m$ is a linear transformation which enforces the moment matrix structure.

Theorem 1 (First step facial reduction) *Suppose $\mathcal{A} : \mathcal{S}^k \to \mathcal{R}^m$ is a linear transformation, $P \in \mathcal{S}^k$, $B \in \mathcal{R}^{k \times l}$, $k > l$. Suppose $\bar{A}_t := V^T A_t V$ and $V \in \mathcal{R}^{k \times d}, k > d, V^T V = I, B^T V = 0, [B\ V]$ nonsingular. Suppose $\bar{\mathcal{A}} : \mathcal{S}^d \to \mathcal{R}^m$ is the linear transformation induced by $\bar{A}_t$. Then*

$$\exists P \in \mathcal{S}^k, \mathcal{A}(P) = b, B^T P = 0, P \succeq 0 \tag{2a}$$

$$\Longleftrightarrow$$

$$\exists \bar{P} \in \mathcal{S}^d, \bar{\mathcal{A}}(\bar{P}) = \bar{b}, \bar{P} \succeq 0. \tag{2b}$$

The following two theorems provide the foundation, the auxiliary problems, for doing facial reduction.

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Theorem 2 *[2, 3] Suppose $\mathcal{A} : \mathcal{S}^k{}_+ \to \mathcal{R}^m$ is a linear transformation, $b \in \mathcal{R}^m$, $P \in \mathcal{S}^k$ and $Z \in \mathcal{S}^k$. Then exactly one of the following alternative systems is consistent.*

$$(I) \qquad 0 \prec P \in F := \{P \in \mathcal{S}^k : \mathcal{A}(P) = b, P \succeq 0\} \tag{3a}$$

$$\iff$$

$$(II) \qquad 0 \neq Z = \mathcal{A}^* y \succeq 0, b^T y = 0. \tag{3b}$$

The algorithm for doing facial reductions can be summarized as follows:

Input($\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m, b \in \mathbb{R}^m$);
repeat
  Solve the auxiliary problem (3b) for $p^*(\mathcal{A}, b)$ by Douglas-Rachford method.
  if $p^*(\mathcal{A}, b) > 0$ **then**
    STOP, facial reduction is done, Slater condition holds
  else
    Apply eigenvalue decomposition to $Z = \bar{\mathcal{A}}^* y$ to obtain W such that $W^T Z = 0$
    Compute $\hat{\mathcal{A}}$ such that $\hat{A}_i \leftarrow W^T \bar{A}_i W, \forall i \in \mathcal{E}$, also update $\hat{b} = \bar{b}$ by removing possibly redundant relations. Record the rank of each face.
  end
until $p^*(\mathcal{A}, b) > 0$;
Solve $\hat{\mathcal{A}}(\hat{P}) = \hat{b}, \hat{P} \succ 0$
Output($\hat{P}$ *which has maximum rank*)

Algorithm 1: facial reduction on the primal.

Theorem 3 (Maximum rank) $p^*(\mathcal{A}, b) > 0$ *if and only if no further facial reductions can be done. Algorithm 1 returns a maximum rank solution of Problem (2a).*

We consider the *Douglas-Rachford reflection-projection* (DR) method which involes projecting and reflections between two convex sets. These two convex sets are the linear constraints and the face in our case. There are also other projection-based methods, such as method of *alternating projection*, we prefer DR method as it displays better convergence properties in our tests. To do that, we need to reformulate the auxiliary problem.

Define s2vec to be the vectorization of a matrix. Suppose A is the matrix form of $\mathcal{A}$, problem (2b) can be converted to

$$\begin{aligned} \text{Find } y \in \mathcal{R}^m : b^T y = 0, A^T y - \text{s2vec}(Z) = 0, \\ Z \succeq 0, \text{trace}(Z) = 1. \end{aligned} \tag{4}$$

which is equivalent to

$$L \cdot \text{s2vec}(Z) = R, Z \succeq 0, \tag{5}$$

Where $L = [b^T \cdot (A^T)^+; I - A^T \cdot (A^T)^+; \text{s2vec}(I)]$ and $R = [0; 0; 1]$.

Given a matrix $X \in \mathcal{S}^k$, we use $\mathcal{P}_{\mathcal{L}}(X)$ to denote the projection onto the linear subspace and use $\mathcal{P}_{\mathcal{F}}(X, \text{rank})$ to denote the projection onto a face with a given rank. One step of DR is :

$$\begin{aligned} Y &= 2 * \mathcal{P}_{\mathcal{F}}(X, \text{rank}) - X, \\ Z &= 2 * \mathcal{P}_{\mathcal{L}}(Y) - Y, \\ X_{new} &= (X + Z)/2. \end{aligned} \tag{6}$$

At each step, we calculate the residual $Res := \|\mathcal{A}(Y) - b\|$, which is the residual with the linear constraints after projecting onto the face. If the residual is less than the given tolerance, we stop and return Y.

1.1 Numerical results

Consider the example $f = \{2yz - y,\ 2y^2 + y,\ xy,\ 4x^2z + 4z^3 + y\}$ in the paper [1].

The kernel of the moment matrix is $\{z^2 + y/2; yz - y/2; y2 + y/2; xz; xy; y + z\}$.

Computing the kernel took 3 facial reductions, rank of each face = [20 16,14,8], toler = [1e-16 1e-15 1e-14]. Residual = 1.49513e-12. Maximum rank is 8.

Compared with SeDuMi(CVX): In SeDuMi(CVX) we have Residual= 0.00284057 and 14 positive eigenvalues with 4 eigenvalues being around 1e-4. Wrong maximum rank!

In Douglas-Rachford iterations, the good choice of rank can improve the accuracy a lot. The strategy to decide the rank is to only count the good eigenvalues (ignore small eigenvalues).

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