Noisy Sensor Network Localization using Semidefinite Representations and Facial Reduction *

Nathan Krislock †   Franz Rendl‡   Henry Wolkowicz§

March 24, 2010

Abstract

In this paper we extend a recent algorithm for solving the sensor network localization problem (***SNL***) to include instances with noisy data. In particular, we continue to exploit the implicit degeneracy in the semidefinite programming (***SDP***) relaxation of ***SNL***. An essential step involves finding good initial estimates for a noisy Euclidean distance matrix, ***EDM***, completion problem. After finding the ***EDM*** completion from the noisy data, we rotate the problem using the original positions of the anchors.

This is a preliminary working paper, and is a work in progress. Tests are currently on-going.

Contents

---

*Department of Combinatorics and Optimization, Waterloo, Ontario N2L 3G1, Canada, Research Report CORR 2010-01.

†Research supported by The Natural Sciences and Engineering Research Council of Canada.

‡Alpe-Adria University Klagenfurt, Institute for Mathematics, Klagenfurt, Austria

§Research supported by The Natural Sciences and Engineering Research Council of Canada.

1 Introduction

In this paper we derive and test an algorithm for solving large scale sensor localization problems (***SNL***) with noisy data. The ***SNL*** problem consists in locating sensors using the fact that some of the sensors are anchors for which the locations are given, and that the distances between sensors within a given *radio range* are approximately known. Our algorithm extends the results in [17], where exact data is assumed.

We initialize the algorithm by finding rigid subgraphs (such as cliques) in the graph corresponding to the ***SNL*** problem. We then localize the positions for each of the subgraphs by treating them as an anchorless ***SNL*** problem, i.e., as a *Euclidean Distance Matrix* (***EDM***) completion problem. This is done using a Newton type method on the corresponding unconstrained minimization problem. Thus we find a good initial estimate for a noisy ***EDM*** .

Following the approach in [17], we then exploit the implicit degeneracy in the semidefinite programming (***SDP***) relaxation of ***SNL*** . This involves finding the subspace representation of the faces of the semidefinite cone $\mathcal{S}^n_+$ corresponding to the faces of the Euclidean distance matrix cone $\mathcal{E}^n$. We repeatedly find the intersection of faces by finding the intersection of the subspace representations. We delay the completion of the original ***EDM*** till the end, i.e., after we find the ***EDM*** completion from the noisy data, we finalize by rotating the problem using the original positions of the anchors.

2 Background

The ***SNL*** problem has recently attracted a lot of interest; see, for example, [7, 19, 21, 22, 13]. See also the webpage www.convexoptimization.com/dattorro/sensor_network_localization.html and the recent thesis [16]. Nie [19] using sums of squares also includes an error analysis. Noisy distances are handled in [4] using a combination of regularization and refinement.

3 Notation and Preliminary Results

We let $\mathcal{M}^{m\times n}$ denote the vector space of $m\times n$ real matrices equipped with the *trace inner product*, $\langle A,B\rangle = \operatorname{trace} A^TB$; let $\mathcal{M}^n := \mathcal{M}^{n\times n}$ and let $\mathcal{S}^n$ denote the subspace of *real symmetric* $n\times n$ *matrices*; $\mathcal{S}^n_+$ and $\mathcal{S}^n_{++}$ denote the cone of positive semidefinite and positive definite matrices, respectively; $A\succeq B$ and $A\succ B$ denote the Löwner partial order, $A-B\in\mathcal{S}^n_+$ and $A-B\in\mathcal{S}^n_{++}$, respectively; $\mathcal{R}(\mathcal{L})$ and $\mathcal{N}(\mathcal{L})$ denote the range space and null space of the linear transformation $\mathcal{L}$, respectively; we let e denote the vector of ones of appropriate dimension; and we use the MATLAB notation $1{:}n = \{1,\ldots,n\}$. For $M\in\mathcal{M}^n$, we let $\mathcal{S}_\Sigma(M) = \frac{1}{2}(M+M^T)\in\mathcal{S}^n$ denote the *sum symmetrization*. Thus, $\mathcal{S}_\Sigma\colon \mathcal{M}^n\to\mathcal{S}^n$ represents the orthogonal projection onto $\mathcal{S}^n$. The adjoint of $\mathcal{S}_\Sigma$ is given by ${\mathcal{S}_\Sigma}^*(S) = S$, for all $S\in\mathcal{S}^n$. For $M\in\mathcal{M}^{mn}$, we let $\mathcal{S}_\Pi(M) = MM^T\in\mathcal{S}^n$ denote the *product symmetrization*.

For a subset S, let $\operatorname{cone}(S)$ denotes the convex cone generated by the set S. A subset $F\subseteq K$

is a *face of the cone* K, denoted $F \trianglelefteq K$, if

$$\left(x, y \in K,\ \frac{1}{2}(x+y) \in F\right) \implies (\operatorname{cone}\{x, y\} \subseteq F).$$

If $F \trianglelefteq K$, but is not equal to K, we write $F \triangleleft K$. If $\{0\} \neq F \triangleleft K$, then F is a *proper face* of K. For $S \subseteq K$, we let face(S) denote the smallest face of K that contains S. A face $F \trianglelefteq K$ is an *exposed face* if it is the intersection of K with a hyperplane. The cone K is *facially exposed* if every face $F \trianglelefteq K$ is exposed.

The cone of positive semidefinite matrices $\mathcal{S}^n_+$ is facially exposed. A face $F \trianglelefteq \mathcal{S}^n_+$ can be characterized using the range or the nullspace of any matrix S in the relative interior of the face. If $S \in \operatorname{relint} F$, and $S = UD_SU^T$ is the compact spectral decomposition of S with the diagonal matrix of eigenvalues $D_S \in \mathcal{S}^t_{++}$, then

$$F = U\mathcal{S}^t_+U^T. \tag{3.1}$$

We let $\mathcal{S}_H \subseteq \mathcal{S}^n$ denote the space of *hollow matrices*; i.e., the set of symmetric matrices with zero diagonal. Let $D \in \mathcal{S}_H$. If there exist points $p_1, \ldots, p_n \in \mathbb{R}^r$ such that

$$D_{ij} = \|p_i - p_j\|_2^2, \quad i, j = 1, \ldots, n, \tag{3.2}$$

then D is called a *Euclidean distance matrix*, denoted $\boldsymbol{EDM}$. Note that we work with *squared distances*. The smallest value of r such that (3.2) holds is called the *embedding dimension* of D. The set of $\boldsymbol{EDM}$ matrices forms a closed convex cone in $\mathcal{S}^n$, denoted $\mathcal{E}^n$. If we are given a partial $\boldsymbol{EDM}$, $D_p \in \mathcal{E}^n$, let $\mathcal{G} = (N, E, \omega)$ be the corresponding simple graph on the nodes $N = 1{:}n$ whose edges E correspond to the known entries of D_p, with $(D_p)_{ij} = \omega_{ij}^2$, for all $(i, j) \in E$.

Definition 3.1. *For $Y \in \mathcal{S}^n$ and $\alpha \subseteq 1{:}n$, we let $Y[\alpha]$ denote the corresponding* principal submatrix *formed from the rows and columns with indices α. If, in addition, $|\alpha| = k$ and $\bar{Y} \in \mathcal{S}^k$ is given, then we define*

$$\mathcal{S}^n(\alpha, \bar{Y}) := \left\{Y \in \mathcal{S}^n : Y[\alpha] = \bar{Y}\right\}, \quad \mathcal{S}^n_+(\alpha, \bar{Y}) := \left\{Y \in \mathcal{S}^n_+ : Y[\alpha] = \bar{Y}\right\}.$$

Definition 3.2. *Given $D \in \mathcal{E}^n$ and $\alpha \subseteq 1{:}n$, let $B := \mathcal{K}^\dagger(D[\alpha]) = P_\alpha P_\alpha^T$, where P is full column rank. Then the rows of P are called a representation of the points in the subset α.*

The subset of matrices in $\mathcal{S}^n$ with the top left $k \times k$ block fixed is

$$\mathcal{S}^n(1{:}k, \bar{Y}) = \left\{Y \in \mathcal{S}^n : Y = \left[\begin{array}{c|c} \bar{Y} & \cdot \\ \hline \cdot & \cdot \end{array}\right]\right\}. \tag{3.3}$$

Similarly, if the principal submatrix $\bar{D} \in \mathcal{E}^k$ is given, for index set $\alpha \subseteq 1{:}n$, with $|\alpha| = k$, we define

$$\mathcal{E}^n(\alpha, \bar{D}) := \left\{D \in \mathcal{E}^n : D[\alpha] = \bar{D}\right\}. \tag{3.4}$$

The subset of matrices in $\mathcal{E}^n$ with the top left $k \times k$ block fixed is

$$\mathcal{E}^n(1{:}k, \bar{D}) = \left\{D \in \mathcal{E}^n : D = \left[\begin{array}{c|c} \bar{D} & \cdot \\ \hline \cdot & \cdot \end{array}\right]\right\}. \tag{3.5}$$

We are given a subset (including the distances between anchors) of the (squared) distances from (3.2). This forms a partial ***EDM***, D_p. We intend to solve the ***EDM*** *completion problem*, i.e. finding the missing entries of D_p to complete the ***EDM*** . This completion problem can be solved by finding a set of points $p_1, \dots, p_n \in \mathbb{R}^r$ satisfying (3.2), where r is the embedding dimension of the partial ***EDM***, D_p. Equivalently, we solve the graph realizability problem with dimension r, i.e. we finding positions in $\mathbb{R}^r$ for the vertices of a graph such that the inter-distances of these positions satisfy the given edge lengths of the graph.

Let $Y \in \mathcal{M}^n$ be an $n \times n$ real matrix and $y \in \mathbb{R}^n$ a vector. We let $\mathrm{diag}(Y)$ denote the vector in $\mathbb{R}^n$ formed from the diagonal of Y. Then $\mathrm{Diag}(y) = \mathrm{diag}^*(y)$ denotes the diagonal matrix in $\mathcal{M}^n$ with the vector y along its diagonal; Diag is the adjoint of diag. The operator offDiag can then be defined as $\mathrm{offDiag}(Y) := Y - \mathrm{Diag}(\mathrm{diag}\, Y)$; let $\mathrm{us2vec} : \mathcal{S}^n \to \mathbb{R}^{n(n-1)/2}$ where $\mathrm{us2vec}(D)$ is $\sqrt{2}$ times the vector in $\mathbb{R}^{n(n-1)/2}$ formed from the strictly upper triangular part of D taken columnwise; the adjoint is $\mathrm{us2Mat} = \mathrm{us2vec}^*$ and $\mathrm{us2Mat}(d) \in \mathcal{S}_H$ takes $\frac{1}{\sqrt{2}}$ times the vector $d \in \mathbb{R}^{n(n-1)/2}$ and forms the matrix in $\mathcal{S}_H$. Note that $\mathrm{us2Mat} = \mathrm{us2vec}^\dagger$, i.e. $\mathrm{us2vec}\,\mathrm{us2Mat} = I$; and us2Mat is an isometry from $\mathbb{R}^{n(n-1)/2}$ to $\mathcal{S}_H$.

For $P^T = \begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix} \in \mathcal{M}^{r \times n}$, where p_j, $j = 1, \dots, n$, are the points used in (3.2), let $Y := PP^T$, and let D be the corresponding ***EDM*** satisfying (3.2). The following linear operators $\mathcal{K}$ and $\mathcal{D}_e$ provide the connection between ***SDP*** and ***EDM*** .

$$\begin{array}{rcl} \mathcal{K}(Y) & := & \mathcal{D}_e(Y) - 2Y \\ & := & \mathrm{diag}(Y)\, e^T + e\ \mathrm{diag}(Y)^T - 2Y \\ & = & \left(p_i^T p_i + p_j^T p_j - 2p_i^T p_j\right)_{i,j=1}^n \\ & = & \left(\|p_i - p_j\|_2^2\right)_{i,j=1}^n \\ & = & D. \end{array} \tag{3.6}$$

By abuse of notation, we also allow $\mathcal{D}_v$ to act on a vector; that is, $\mathcal{D}_v(y) := yv^T + vy^T$. Note that

$$\mathcal{K}(Y) = 2\,\mathcal{S}_\Sigma(\mathrm{diag}(Y)e^T) - 2Y = 2(\mathcal{S}_\Sigma(\cdot e^T)\,\mathrm{diag})(Y) - 2Y. \tag{3.7}$$

Therefore, the adjoint of $\mathcal{K}$ acting on $D \in \mathcal{S}^n$ is

$$\begin{array}{rcl} \mathcal{K}^*(D) & = & 2(\mathcal{S}_\Sigma(\cdot e^T)\,\mathrm{diag})^*(D) - 2D \\ & = & 2\,\mathrm{diag}^*(\cdot e^T)^*(\mathcal{S}_\Sigma)^*(D) - 2D \\ & = & 2\,\mathrm{Diag}(\cdot e)(D) - 2D \\ & = & 2(\mathrm{Diag}(De) - D). \end{array} \tag{3.8}$$

Moreover,

$$\begin{array}{rcl} \mathcal{K}^*\,\mathcal{K}(Y) & = & 2(\mathrm{Diag}(\mathcal{K}(Y)e) - \mathcal{K}(Y)) \\ & = & -2(\mathcal{K}(Y) - \mathrm{Diag}(v)), \end{array} \tag{3.9}$$

where $v = \mathcal{K}(Y)e$, i.e. we write it this way to emphasize that we simply subtract the row sums from the diagonal of $\mathcal{K}(Y)$.

The linear operator $\mathcal{K}$ is one-one and onto between the *centered* and *hollow* subspaces of $\mathcal{S}^n$, which are defined as

$$\begin{array}{rcll} \mathcal{S}_C & := & \{Y \in \mathcal{S}^n : Ye = 0\} & \text{(zero row sums)}, \\ \mathcal{S}_H & := & \{D \in \mathcal{S}^n : \mathrm{diag}(D) = 0\} & = \mathcal{R}(\mathrm{offDiag}). \end{array} \tag{3.10}$$

Let $J := I - \frac{1}{n}ee^T$ denote the orthogonal projection onto the subspace $\{e\}^\perp$ and define the linear operator $\mathcal{T}(D) := -\frac{1}{2}J\,\mathrm{offDiag}(D)J$. Then we have the following relationships.

Proposition 3.3. *([1]) The linear operator $\mathcal{T}$ is the generalized inverse of the linear operator $\mathcal{K}$; that is, $\mathcal{K}^\dagger = \mathcal{T}$. Moreover:*

$$\begin{array}{rl} \mathcal{R}(\mathcal{K}) = \mathcal{S}_H; & \mathcal{N}(\mathcal{K}) = \mathcal{R}(\mathcal{D}_e); \\ \mathcal{R}(\mathcal{K}^*) = \mathcal{R}(\mathcal{T}) = \mathcal{S}_C; & \mathcal{N}(\mathcal{K}^*) = \mathcal{N}(\mathcal{T}) = \mathcal{R}(\mathrm{Diag}); \end{array} \tag{3.11}$$

$$\mathcal{S}^n = \mathcal{S}_H \oplus \mathcal{R}(\mathrm{Diag}) = \mathcal{S}_C \oplus \mathcal{R}(\mathcal{D}_e). \tag{3.12}$$

Theorem 3.4. *([1]) The linear operators $\mathcal{T}$ and $\mathcal{K}$ are one-to-one and onto mappings between the cone $\mathcal{E}^n \subset \mathcal{S}_H$ and the face of the semidefinite cone $\mathcal{S}^n_+ \cap \mathcal{S}_C$. That is,*

$$\mathcal{T}(\mathcal{E}^n) = \mathcal{S}^n_+ \cap \mathcal{S}_C \quad \textit{and} \quad \mathcal{K}(\mathcal{S}^n_+ \cap \mathcal{S}_C) = \mathcal{E}^n.$$

Let $D_p \in \mathcal{S}^n$ be a *partial* ***EDM*** with embedding dimension r and let $H \in \mathcal{S}^n$ be the 0–1 matrix corresponding to the known entries of D_p. One can use the substitution $D = \mathcal{K}(Y)$, where $Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C$, in the ***EDM*** completion problem

$$\begin{array}{cc} \text{Find} & D \in \mathcal{E}^n \\ \text{s.t.} & H \circ D = D_p \end{array}$$

to obtain the ***SDP*** relaxation

$$\begin{array}{cc} \text{Find} & Y \in \mathcal{S}^n_+ \cap \mathcal{S}_C \\ \text{s.t.} & H \circ \mathcal{K}(Y) = D_p \end{array}.$$

This relaxation does not restrict the rank of Y and may yield a solution with embedding dimension that is too large, if $\mathrm{rank}\,(Y) > r$. A clique $\gamma \subseteq 1{:}n$ in the graph $\mathcal{G}$ corresponds to a subset of sensors for which the distances $\omega_{ij} = \|p_i - p_j\|_2$ are known, for all $i, j \in \gamma$; equivalently, the clique corresponds to the principal submatrix $D_p[\gamma]$ of the partial ***EDM*** matrix D_p, where all the elements of $D_p[\gamma]$ are known. Moreover, solving ***SDP*** problems with rank restrictions is NP-HARD. However, we work on faces of $\mathcal{S}^n_+$ described by $U\mathcal{S}^t_+U^T$, with $t \leq n$. In order to find the face with the smallest dimension t, we must have the correct knowledge of the matrix U. In this paper, we obtain information on U using the cliques in the graph of the partial ***EDM***.

Suppose that

$$V^T e = 0 \text{ and } \begin{bmatrix} e & V \end{bmatrix} \text{ is nonsingular.} \tag{3.13}$$

We now introduce the composite operators

$$\mathcal{K}_V(X) \quad := \quad \mathcal{K}(VXV^T), \tag{3.14}$$

and

$$\mathcal{T}_V(D) \quad := \quad V^\dagger \mathcal{T}(D)(V^T)^\dagger = -\tfrac{1}{2} V^\dagger J \,\mathrm{offDiag}(D) J (V^T)^\dagger. \tag{3.15}$$

Lemma 3.5 ([2, 1])**.** *Suppose that V satisfies the definition in (3.13). Then*

$$\begin{array}{c} \mathcal{K}_V(\mathcal{S}_{n-1}) = \mathcal{S}_H, \\ \mathcal{T}_V(\mathcal{S}_H) = \mathcal{S}_{n-1}, \end{array}$$

and $\mathcal{K}_V = \mathcal{T}_V^\dagger$.

From (3.13) and (3.6) we get that

$$\mathcal{K}_V^*(D) \quad = \quad V^T \mathcal{K}^*(D) V \tag{3.16}$$

is the adjoint operator of $\mathcal{K}_V$. The following corollary summarizes useful relationships between $\mathcal{E}$, the cone of Euclidean distance matrices of order n, and $\mathcal{P}$, the cone of positive semidefinite matrices of order $n-1$.

Corollary 3.6 ([2, 1])**.** *Suppose that V is defined as in (3.13). Then:*

$$\begin{aligned} \mathcal{K}_V(\mathcal{P}) &= \mathcal{E}, \\ \mathcal{T}_V(\mathcal{E}) &= \mathcal{P}. \end{aligned}$$

4 Clique Reduction

We now present several techniques for reducing an $\boldsymbol{EDM}$ completion problem using cliques in the graph. This extends the results presented in [8, 9, 17]. In particular, we modify the approach in [17] for combining two cliques.

The following two technical lemmas are given in [17].

Lemma 4.1 ([17])**.** *Let $B \in \mathcal{S}^n$, $Bv = 0$, $v \neq 0$, $y \in \mathbb{R}^n$ and $\bar{Y} := B + \mathcal{D}_v(y)$. If $\bar{Y} \succeq 0$, then*

$$y \in \mathcal{R}(B) + \operatorname{cone}\{v\}.$$

□

Lemma 4.2 ([17])**.** *Let $Y \in \mathcal{S}^k_+$ and $\bar{U} \in \mathcal{M}^{k \times t}$ with $\bar{U}$ having full column rank. If* face $\{\bar{Y}\} \trianglelefteq$ *(resp.=) $\bar{U}\mathcal{S}^t_+\bar{U}^T$, then*

$$\operatorname{face} \mathcal{S}^n_+(1{:}k, \bar{Y}) \trianglelefteq \ \textit{(resp.=)} \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \mathcal{S}^{n-k+t}_+ \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix}^T. \tag{4.1}$$

Proof. The result in [17] assumes that $\bar{U}^T\bar{U} = I$. The extension to $\bar{U}$ having full column rank follows from taking the compact QR-factorization $U = QR$, where $Q^TQ = I$ and R is nonsingular. □

We can now find an expression for the face defined by a given clique in the graph. Without loss of generality, we can assume that $\alpha = 1{:}k \subseteq 1{:}n$, $|\alpha| = k$.

Theorem 4.3 ([17])**.** *Let $D \in \mathcal{E}^n$, with embedding dimension r. Let $\alpha := 1{:}k$, $\bar{D} := D[\alpha] \in \mathcal{E}^k$ with embedding dimension t, and $B := \mathcal{K}^\dagger(\bar{D}) = \bar{U}_B S \bar{U}_B^T$, where $\bar{U}_B \in \mathcal{M}^{k \times t}$, $\bar{U}_B$ having full column rank, and $S \in \mathcal{S}^t_{++}$. Furthermore, let $U_B := \begin{bmatrix} \bar{U}_B & \frac{1}{\sqrt{k}}e \end{bmatrix} \in \mathcal{M}^{k \times (t+1)}$, $U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}$, and let $\begin{bmatrix} V & \frac{U^Te}{\|U^Te\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal. Then*

$$\operatorname{face} \mathcal{K}^\dagger\left(\mathcal{E}^n(1{:}k, \bar{D})\right) = \left(U\mathcal{S}^{n-k+t+1}_+ U^T\right) \cap \mathcal{S}_C = (UV)\mathcal{S}^{n-k+t}_+(UV)^T. \tag{4.2}$$

Proof. As in Lemma 4.2, the result in [17] assumes that $\bar{U}^T\bar{U} = I$. The extension follows as in the proof of the Lemma. □

In Theorem 4.3 we can make various choices for S and thus change the choice of $\bar{U}_B$. An interesting choice for $\bar{U}_B$ allows for a representation for the points in the clique.

Corollary 4.4. *Let $D, r, \alpha, \bar{D}, t$ be defined as in Theorem 4.3. Let $B := \mathcal{K}^\dagger(\bar{D}) = P_B P_B^T$, where $P_B \in \mathcal{M}^{k\times t}$ is full column rank. Furthermore, let Q be orthogonal, $U_B := \begin{bmatrix} P_B Q & \frac{1}{\sqrt{k}}e \end{bmatrix} \in \mathcal{M}^{k\times(t+1)}$, $U := \begin{bmatrix} U_B & 0 \\ 0 & I_{n-k} \end{bmatrix}$, and let $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal. Then (4.2) holds, and the rows of $P_B Q$ provide a relative representation of the points in the clique α, i.e.*

$$\mathcal{K}((P_B Q)(P_B Q)^T) = D[\alpha].$$

Proof. We just need to use $S = I_t = QQ^T$ in the expression for B in the hypothesis of Theorem 4.3; e.g. we could use the compact spectral decomposition $B = UDU^T$ and set $P_B = UD^{1/2}$. Then $\mathcal{K}((P_B Q)(P_B Q)^T) = \mathcal{K}(P_B(P_B^T) = \mathcal{K}(B) = D[\alpha]$. $\square$

The following result provides expressions for the face for the union of two cliques.

Theorem 4.5 ([17])**.** *Let $D \in \mathcal{E}^n$ with embedding dimension r and define the sets of positive integers*

$$\begin{array}{c} \alpha_1 := 1{:}(\bar{k}_1 + \bar{k}_2), \quad \alpha_2 := (\bar{k}_1 + 1){:}(\bar{k}_1 + \bar{k}_2 + \bar{k}_3) \subseteq 1{:}n, \\ k_1 := |\alpha_1| = \bar{k}_1 + \bar{k}_2, \quad k_2 := |\alpha_2| = \bar{k}_2 + \bar{k}_3, \\ k := \bar{k}_1 + \bar{k}_2 + \bar{k}_3. \end{array} \tag{4.3}$$

For $i = 1, 2$, let $\bar{D}_i := D[\alpha_i] \in \mathcal{E}^{k_i}$ with embedding dimension t_i, and $B_i := \mathcal{K}^\dagger(\bar{D}_i) = \bar{U}_i S_i \bar{U}_i^T$, where $\bar{U}_i \in \mathcal{M}^{k_i\times t_i}$, $\bar{U}_i^T \bar{U}_i = I_{t_i}$, $S_i \in \mathcal{S}_{++}^{t_i}$, and $U_i := \begin{bmatrix} \bar{U}_i & \frac{1}{\sqrt{k_i}}e \end{bmatrix} \in \mathcal{M}^{k_i\times(t_i+1)}$. Let t and $\bar{U} \in \mathcal{M}^{k\times(t+1)}$ satisfy

$$\mathcal{R}(\bar{U}) = \mathcal{R}\left(\begin{bmatrix} U_1 & 0 \\ 0 & I_{\bar{k}_3} \end{bmatrix}\right) \cap \mathcal{R}\left(\begin{bmatrix} I_{\bar{k}_1} & 0 \\ 0 & U_2 \end{bmatrix}\right), \text{ with } \bar{U}^T\bar{U} = I_{t+1}. \tag{4.4}$$

Let $U := \begin{bmatrix} \bar{U} & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathcal{M}^{n\times(n-k+t+1)}$ and $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix} \in \mathcal{M}^{n-k+t+1}$ be orthogonal. Then

$$\bigcap_{i=1}^{2} \text{face}\, \mathcal{K}^\dagger\left(\mathcal{E}^n(\alpha_i, \bar{D}_i)\right) = \left(U\mathcal{S}_+^{n-k+t+1}U^T\right) \cap \mathcal{S}_C = (UV)\mathcal{S}_+^{n-k+t}(UV)^T. \tag{4.5}$$

$\square$

4.1 Nonsingular Reduction with Intersection Embedding Dimension r

We need the following technical result on the intersection of two structured subspaces.

Lemma 4.6 ([17])**.** *Let*

$$U_1 := \begin{bmatrix} U_1' \\ U_1'' \end{bmatrix}, \quad U_2 := \begin{bmatrix} U_2'' \\ U_2' \end{bmatrix}, \quad \hat{U}_1 := \begin{bmatrix} U_1' & 0 \\ U_1'' & 0 \\ 0 & I \end{bmatrix}, \quad \hat{U}_2 := \begin{bmatrix} I & 0 \\ 0 & U_2'' \\ 0 & U_2' \end{bmatrix}$$

be appropriately blocked with $U_1'', U_2'' \in \mathcal{M}^{k\times l}$ *full column rank and* $\mathcal{R}(U_1'') = \mathcal{R}(U_2'')$*. Furthermore, let*

$$\bar{U}_1 := \begin{bmatrix} U_1' \\ U_1'' \\ U_2'(U_2'')^\dagger U_1'' \end{bmatrix}, \quad \bar{U}_2 := \begin{bmatrix} U_1'(U_1'')^\dagger U_2'' \\ U_2'' \\ U_2' \end{bmatrix}. \tag{4.6}$$

Then $\bar{U}_1$ *and* $\bar{U}_2$ *are full column rank and satisfy*

$$\mathcal{R}(\hat{U}_1) \cap \mathcal{R}(\hat{U}_2) = \mathcal{R}\left(\bar{U}_1\right) = \mathcal{R}\left(\bar{U}_2\right).$$

Moreover, if $e_l \in \mathbb{R}^l$ *is the* l^{th} *standard unit vector, and* $U_i e_l = \alpha_i e$*, for some* $\alpha_i \neq 0$*, for* $i = 1, 2$*, then* $\bar{U}_i e_l = \alpha_i e$*, for* $i = 1, 2$*.* □

The following key result shows that we can complete the distances in the union of two cliques provided that their intersection has embedding dimension equal to r.

Theorem 4.7 ([17])**.** *Let the hypotheses of Theorem 4.5 hold. Let*

$$\beta \subseteq \alpha_1 \cap \alpha_2, \quad \bar{D} := D[\beta], \quad B := \mathcal{K}^\dagger(\bar{D}), \quad \bar{U}_\beta := \bar{U}[\beta, :],$$

where $\bar{U} \in \mathcal{M}^{k\times(t+1)}$ *satisfies equation (4.4). Let* $\begin{bmatrix} \bar{V} & \frac{\bar{U}^T e}{\|\bar{U}^T e\|} \end{bmatrix} \in \mathcal{M}^{t+1}$ *be orthogonal. Let*

$$Z := (J\bar{U}_\beta \bar{V})^\dagger B((J\bar{U}_\beta \bar{V})^\dagger)^T. \tag{4.7}$$

If the embedding dimension for $\bar{D}$ *is* r*, then* $t = r$*,* $Z \in \mathcal{S}^r_{++}$ *is the unique solution of the equation*

$$(J\bar{U}_\beta \bar{V})Z(J\bar{U}_\beta \bar{V})^T = B, \tag{4.8}$$

and

$$D[\alpha_1 \cup \alpha_2] = \mathcal{K}\left((\bar{U}\bar{V})Z(\bar{U}\bar{V})^T\right). \tag{4.9}$$

□

The following result shows that if we know the minimal face of $\mathcal{S}^n_+$ containing $\mathcal{K}^\dagger(D)$, and we know a small submatrix of D, then we can compute a set of points in $\mathbb{R}^r$ that generate D by solving a small equation.

Corollary 4.8 ([17])**.** *Let* $D \in \mathcal{E}^n$ *with embedding dimension* r*, and let* $\beta \subseteq 1{:}n$*. Let* $U \in \mathcal{M}^{n\times(r+1)}$ *satisfy*

$$\operatorname{face} \mathcal{K}^\dagger(D) = \left(U\mathcal{S}^{r+1}_+ U^T\right) \cap \mathcal{S}_C,$$

let $U_\beta := U[\beta, :]$*, and let* $\begin{bmatrix} V & \frac{U^T e}{\|U^T e\|} \end{bmatrix} \in \mathcal{M}^{r+1}$ *be orthogonal. If* $D[\beta]$ *has embedding dimension* r*, then*

$$(JU_\beta V)Z(JU_\beta V)^T = \mathcal{K}^\dagger(D[\beta])$$

has a unique solution $Z \in \mathcal{S}^r_{++}$*, and* $D = \mathcal{K}(PP^T)$*, where* $P := UVZ^{1/2} \in \mathbb{R}^{n\times r}$*.* □

We now show that we can combine two cliques using the relative point representations of each.

Theorem 4.9. *Let the hypotheses of Theorem 4.5 hold, and following Corollary 4.4, for* $i = 1, 2$, *let* $B_i = P_i P_i^T$ *be full column rank factorizations, so that the rows of* P_i *provide relative positions for the points in the cliques* α_i*; and partition*

$$P_1 := \begin{bmatrix} P_1' \\ P_1'' \end{bmatrix}, \quad P_2 := \begin{bmatrix} P_2'' \\ P_2' \end{bmatrix}, \quad \hat{P}_1 := \begin{bmatrix} P_1' & 0 \\ P_1'' & 0 \\ 0 & I \end{bmatrix}, \quad \hat{P}_2 := \begin{bmatrix} I & 0 \\ 0 & P_2'' \\ 0 & P_2' \end{bmatrix}.$$

Furthermore, let

$$\bar{P}_1 := \begin{bmatrix} P_1' \\ P_1'' \\ P_2'(P_2'')^\dagger P_1'' \end{bmatrix}, \quad \bar{P}_2 := \begin{bmatrix} P_1'(P_1'')^\dagger P_2'' \\ P_2'' \\ P_2' \end{bmatrix}. \tag{4.10}$$

If the embedding dimension of $\bar{D}$ *is* r*, then:* $t = r$*;* $Q_1 := (P_1'')^\dagger P_2''$ *and* $Q_2 := (P_2'')^\dagger P_1''$ *are both orthogonal;* $\bar{P}_1$ *and* $\bar{P}_2$ *are full column rank and their rows provide relative representations for the points in the union of the cliques* $\alpha_i, i = 1, 2$*, i.e.*

$$D[\alpha_1 \cup \alpha_2] = \mathcal{K}(\bar{P}_i \bar{P}_i^T), \quad i = 1, 2. \tag{4.11}$$

Proof. From Lemma 4.6, we have that $\mathcal{R}(\bar{P}_1) = \mathcal{R}(\bar{P}_2)$. Therefore, $\mathcal{R}(P_1'') = \mathcal{R}(P_2'')$. This means that we can apply the projections on these ranges and get that

$$P_2''(P_2'')^\dagger P_1'' = P_1''; \quad P_1''(P_1'')^\dagger P_2'' = P_2''.$$

Therefore, $\bar{P}_1$ is obtained using $Q_1 = (P_1'')^\dagger P_2''$ and the multiplication $P_1 Q_1$. Similarly, $\bar{P}_2$ is obtained using $Q_2 = (P_2'')^\dagger P_1''$ and the multiplication $P_2 Q_2$.

Since

$$P_i e = 0, \quad \bar{D} = \mathcal{K}(P_i''(P_i'')^T), \qquad i = 1, 2,$$

We get that both $Q_i, i = 1, 2$ are orthogonal. □

Remark 4.10. *Note that there can be many ways to find the full column rank factorizations* $B_i = P_i P_i^T$ *in Theorem 4.9, e.g.: the compact spectral decomposition; the partial Cholesky factorization; or the compact* QR *factorization.*

5 Nearest EDM

Suppose that we have a clique α corresponding to the ***EDM*** $\bar{D}$. Then we can find the smallest face containing $\mathcal{E}^n(\alpha, \bar{D})$ using $B = \mathcal{K}^\dagger(\bar{D})$; see [17]. We now consider the case when we are given a possibly noisy ***EDM*** and we would like to find a best approximation, or the nearest ***EDM*** , i.e. we want to find a best approximation of $B = \mathcal{K}^\dagger(\bar{D})$ with the correct rank r. For this purpose, we let $D \in \mathcal{E}^n$ with embedding dimension r, and suppose that $D_\epsilon = D + N_\epsilon \in \mathcal{S}_H \cap \mathcal{N}$, where the off diagonal elements of the rows of the error matrix $N_\epsilon \in \mathcal{S}_H$ are independently and identically distributed with zero mean and the same variance. We now look for the best approximation to the given noisy distance matrix $D_\epsilon (= \bar{D})$. For example, we could do this in two steps: we first find the least squares solution $B_\epsilon = \mathcal{K}^\dagger(D_\epsilon)$, which may not be positive semidefinite and may have the wrong rank; we then use the truncated spectral decomposition $B_\epsilon = \mathcal{K}^\dagger(D_\epsilon) \approx U_\epsilon \Sigma_\epsilon U_\epsilon^T$, where $U_\epsilon^T U_\epsilon = I_r$, $\Sigma_\epsilon \in \mathcal{S}^r_{++}$. We could then use the approximation $\mathcal{K}(U_\epsilon \Sigma_\epsilon U_\epsilon^T) \approx D_\epsilon$, i.e.

$$D_\epsilon \approx \mathcal{K}(U_\epsilon \Sigma_\epsilon U_\epsilon^T), \qquad \text{where } B_\epsilon = \mathcal{K}^\dagger(D_\epsilon) \approx U_\epsilon \Sigma_\epsilon U_\epsilon^T,\ U_\epsilon^T U_\epsilon = I_r, \text{ and } \Sigma_\epsilon \in \mathcal{S}^r_{++}. \tag{5.1}$$

Alternatively, we could solve the $\boldsymbol{SDP}$

$$\begin{array}{ll} \min & \|\mathcal{K}(X) - D_\epsilon\| \\ \text{s.t.} & \operatorname{rank}(X) = r \\ & Xe = 0 \\ & X \succeq 0. \end{array}$$

We could introduce V as in Corollary 3.6, eliminate the constraints. We get the following.

Problem 5.0.1. *The unconstrained nearest* $\boldsymbol{EDM}$ *problem with embedding dimension* r *is*

$$\begin{array}{rcll} U_r^* & \in & \operatorname{argmin} & \frac{1}{2}\left\|\mathcal{K}_V(UU^T) - D_\epsilon\right\|_F^2 \\ & & s.t. & U \in M^{(n-1)r}. \end{array} \tag{5.2}$$

The nearest $\boldsymbol{EDM}$ *is* $D^* = \mathcal{K}_V(U_r^*(U_r^*)^T)$.

5.1 Noise model

Let $D \in \mathcal{E}^n$ with embedding dimension r and suppose that $D_\epsilon = D + N_\epsilon$. We assume that there is a multiplicative error on the measured distances,

$$\hat{d}_{ij} = \|p_i - p_j\|_2(1 + \sigma\epsilon_{ij}),$$

where $\sigma \in [0, 1]$ is the noise factor, and ϵ_{ij} are independently and identically distributed random variables coming from the normal distribution with mean zero and variance one. Then

$$D_\epsilon = \left(\hat{d}_{ij}^2\right) = \left(\|p_i - p_j\|_2^2(1 + \sigma\epsilon_{ij})^2\right),$$

and $D = \left(\|p_i - p_j\|_2^2\right)$, so that

$$N_\epsilon = D_\epsilon - D = \left(\|p_i - p_j\|_2^2(2\sigma\epsilon_{ij} + \sigma^2\epsilon_{ij}^2)\right).$$

We used the *multiplicative noise model* noise model for our tests on noisy problems:

$$d_{ij} = \|p_i - p_j\|(1 + \sigma\varepsilon_{ij}), \quad \text{for all } ij \in E,$$

where $\sigma \geq 0$ represents the *noise factor* and, for all $ij \in E$, the random variable ε_{ij} is normally distributed with zero mean and standard deviation one. That is, $\{\varepsilon_{ij}\}_{ij\in E}$ are uncorrelated, have zero mean and the same variance. Here we are modelling the situation that the amount of additive noise corrupting a distance measurement between two sensors is directly proportional to the distance between the sensors.

This multiplicative noise model is the one most commonly considered in sensor network localization; see, for example, [6],[5],[23],[24],[20],[14, 15]. For large values of σ, it is possible that $1+\sigma\varepsilon$ is negative. Therefore, the alternate multiplicative noise model

$$d_{ij} = \|p_i - p_j\||1 + \sigma\varepsilon_{ij}|, \quad \text{for all } ij \in E,$$

is sometimes used. Note, however, that in both multiplicative noise models, we have

$$d_{ij}^2 = \|p_i - p_j\|^2(1 + \sigma\varepsilon_{ij})^2, \quad \text{for all } ij \in E. \tag{5.3}$$

The associated least squares problem for determining the *maximum likelihood* positions for $p_1, \ldots, p_n \in \mathbb{R}^r$ is

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{ij \in E} v_{ij}^2 \\ \text{subject to} & \|p_i - p_j\|^2 (1 + v_{ij})^2 = d_{ij}^2, \quad \text{for all } ij \in E \\ & \sum_{i=1}^n p_i = 0 \\ & p_1, \ldots, p_n \in \mathbb{R}^r. \end{array}$$

Let H be the 0–1 adjacency matrix associated with the n-by-n partial Euclidean distance matrix $D := (d_{ij}^2)$. Letting $V := (v_{ij}) \in \mathbb{R}^{n \times n}$ and $V_H := H \circ V$, we can rewrite this problem as

$$\begin{array}{ll} \text{minimize} & \|V_H\|_F^2 \\ \text{subject to} & \mathcal{K}(PP^T) \circ (H + 2V_H + V_H \circ V_H) = H \circ D \\ & P^T e = 0 \\ & P \in \mathbb{R}^{n \times r}. \end{array}$$

Removing the rank constraint, we obtain the (nonlinear) semidefinite relaxation

$$\begin{array}{ll} \text{minimize} & \|V_H\|_F^2 \\ \text{subject to} & \mathcal{K}(Y) \circ (H + 2V_H + V_H \circ V_H) = H \circ D \\ & Y \in \mathcal{S}_+^n \cap \mathcal{S}_C. \end{array} \tag{5.4}$$

We compare the following two approaches. Let D be an n-by-n Euclidean distance matrix corrupted by noise (hence D may not even be a true Euclidean distance matrix since $\mathcal{K}^\dagger(D)$ may have negative eigenvalues).

1. We compute the eigenvalue decomposition $\mathcal{K}^\dagger(D) = U \Lambda U^T$, and let $P := U_r \Lambda_r^{1/2} \in \mathbb{R}^{n \times r}$. This matrix P minimizes $\left\|PP^T - \mathcal{K}^\dagger(D)\right\|_F$ over all $P \in \mathbb{R}^{n \times r}$, and satisfies

   $$\left\|PP^T - \mathcal{K}^\dagger(D)\right\|_F = \sqrt{\sum_{i=r+1}^n \lambda_i^2(\mathcal{K}^\dagger(D))}.$$

   Since we assume that $\operatorname{diag}(D) = 0$, we have that $\mathcal{K}\mathcal{K}^\dagger(D) = D$. Therefore,

   $$\begin{aligned} \left\|\mathcal{K}(PP^T) - D\right\|_F &= \left\|\mathcal{K}(PP^T - \mathcal{K}^\dagger(D))\right\|_F \\ &\le \|\mathcal{K}\|_F \cdot \left\|PP^T - \mathcal{K}^\dagger(D)\right\|_F \\ &= 2\sqrt{n}\sqrt{\sum_{i=r+1}^n \lambda_i^2(\mathcal{K}^\dagger(D))}. \end{aligned}$$

2. We can compute better Euclidean distance matrix approximations of D by increasing the rank of the approximation PP^T of $\mathcal{K}^\dagger(D)$. That is, we let $P := U_k \Lambda_k^{1/2} \in \mathbb{R}^{n \times k}$, for some $k > r$; see, for example, [2, Lemma 2]. Our facial reduction technique lends well to this approach. There is no problem for us to compute the intersection of different faces that occupy different dimensions.

6 Clique Reductions Algorithm

In [17], we presented an algorithm for exact ***SNL*** given exact data. We now outline this algorithm and extend to include new cases. The algorithm in [17] considered/handled four different cases:

1. Rigid clique intersection:
2. Non-rigid clique intersection:
3. Rigid node absorption:
4. Non-rigid node absorption:

Each of these cases made use of the essential fact that we knew the embedding dimension is r and that the operation resulted in a unique face of dimension r from the intersection of two faces. This resulted in a very successful algorithm. However, the algorithm could fail when the graph is very sparse. This is due to the fact that the intersection process did not result in a unique face of proper dimension.

In the case that we end up with more than one clique after applying the above four techniques, we now extend it to allow the intersection of faces to have dimension $> r$. For example, the singular intersections with the application of lower bounds may not yield a unique solution, so we let the solution be in a higher dimension. This still reduces the dimension of the current face of the problem.

7 Generating/Testing Instances

The ***SNL*** is closely related to the molecular distance geometry problem; see, for example, [12, 11, 10, 3]. In particular, the *Extended Geometric Build-up Algorithm, (EGBA)* is presented in [12]. This algorithm, for $r = 3$ starts with a clique made up of 4 atoms, and then builds up the size of the clique by adding one atom at a time. They include a discussion on how to avoid the build up of round-off error. See [18] for information on generating instances for the molecular distance geometry problem.

References

[1] S. AL-HOMIDAN and H. WOLKOWICZ. Approximate and exact completion problems for Euclidean distance matrices using semidefinite programming. *Linear Algebra Appl.*, 406:109–141, 2005. 5, 6

[2] A. ALFAKIH, A. KHANDANI, and H. WOLKOWICZ. Solving Euclidean distance matrix completion problems via semidefinite programming. *Comput. Optim. Appl.*, 12(1-3):13–30, 1999. Computational optimization—a tribute to Olvi Mangasarian, Part I. 5, 6, 11

[3] L.T.H. AN and P.D. TAO. Large scale molecular conformation via the exact distance geometry problem. In *Optimization (Namur, 1998)*, volume 481 of *Lecture Notes in Econom. and Math. Systems*, pages 260–277. Springer, Berlin, 2000. 12

[4] P. BISWAS, T.-C. LIANG, Y. YE, K-C. TOH, and T.-C. WANG. Semidefinite programming approaches for sensor network localization with noisy distance measurements. *IEEE Transactions onAutomation Science and Engineering*, 3(4):360–371, October 2006. 2

[5] P. BISWAS, T.C. LIANG, K.C. TOH, T.C. WANG, and Y. YE. Semidefinite programming approaches for sensor network localization with noisy distance measurements. *IEEE Transactions on Automation Science and Engineering*, 3:360–371, 2006. 10

[6] P. BISWAS and Y. YE. Semidefinite programming for ad hoc wireless sensor network localization. In *Information Processing In Sensor Networks, Proceedings of the third international symposium on Information processing in sensor networks*, pages 46–54, Berkeley, Calif., 2004. 10

[7] M.W. CARTER, H.H. JIN, M.A. SAUNDERS, and Y. YE. SpaseLoc: an adaptive subproblem algorithm for scalable wireless sensor network localization. *SIAM J. Optim.*, 17(4):1102–1128, 2006. 2

[8] Y. DING, N. KRISLOCK, J. QIAN, and H. WOLKOWICZ. Sensor network localization, Euclidean distance matrix completions, and graph realization. *Optimization and Engineering*, to appear(CORR 2006-23, to appear), 2006. 6

[9] Y. DING, N. KRISLOCK, J. QIAN, and H. WOLKOWICZ. Sensor network localization, Euclidean distance matrix completions, and graph realization. In *Proceedings of MELT08, San Francisco*, pages 129–134, 2008. 6

[10] Q. DONG and Z. WU. A linear-time algorithm for solving the molecular distance geometry problem with exact inter-atomic distances. *J. Global Optim.*, 22(1-4):365–375, 2002. Dedicated to Professor Reiner Horst on his 60th birthday. 12

[11] Q. DONG and Z. WU. A geometric build-up algorithm for solving the molecular distance geometry problem with sparse distance data. *J. Global Optim.*, 26(3):321–333, 2003. 12

[12] R. DOS SANTOS CARVALHO, C. LAVOR, and F. PROTTI. Extending the geometric build-up algorithm for the molecular distance geometry problem. *Inform. Process. Lett.*, 108(4):234–237, 2008. 12

[13] S. KIM, M. KOJIMA, and H. WAKI. Exploiting sparsity in SDP relaxation for sensor network localization. *SIAM J. Optim.*, 20(1):192–215, 2009. 2

[14] S. KIM, M. KOJIMA, and H. WAKI. Exploiting sparsity in SDP relaxation for sensor network localization. *SIAM J. Optim.*, 20(1):192–215, 2009. 10

[15] S. KIM, M. KOJIMA, H. WAKI, and M. YAMASHITA. A sparse version of full semidefinite programming relaxation for sensor network localization problems. Technical Report B-457, Department of Mathematical and Computing Sciences Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo 152-8552, 2009. 10

[16] N. KRISLOCK. *Efficient algorithms for large-scale Euclidean distance matrix problems with applications to wireless sensor network localization and molecular conformation.* PhD thesis, University of Waterloo, 2010. 2

[17] N. KRISLOCK and H. WOLKOWICZ. Explicit sensor network localization using semidefinite representations and clique reductions. Technical Report CORR 2009-04, University of Waterloo, Waterloo, Ontario, 2009. Available at URL: `www.optimization-online.org/DB_HTML/2009/05/2297.html`. 2, 6, 7, 8, 9, 12

[18] C. LAVOR. On generating instances for the molecular distance geometry problem. In *Global optimization*, volume 84 of *Nonconvex Optim. Appl.*, pages 405–414. Springer, New York, 2006. 12

[19] J. NIE. Sum of squares method for sensor network localization. *Comput. Optim. Appl.*, 43:151–179, 2009. 2

[20] T.K. PONG and P. TSENG. (Robust) edge-based semidefinite programming relaxation of sensor network localization. Technical Report Jan-09, University of Washington, Seattle, WA, 2009. 10

[21] C. SAVARESE, J. RABAEY, , and J. BEUTEL. Locationing in distributed ad-hoc wireless sensor networks. In *IEEE Int. Conf. on Acoustics, Speech, and Signal Processing (ICASSP)*, pages 2037–2040, 2001. 2

[22] A. SAVVIDES, C.C. HAN, and M.B. SRIVASTAVA. Dynamic fine grained localization in ad-hoc sensor networks. In *Proceedings of the Fifth International Conference on Mobile Computing and Networking (Mobicom 2001)*, pages 166–179, 2001. 2

[23] P. TSENG. Second-order cone programming relaxation of sensor network localization. *SIAM J. Optim.*, 18(1):156–185, 2007. 10

[24] Z. WANG, S. ZHENG, S. BOYD, and Y. YE. Further relaxations of the semidefinite programming approach to sensor network localization. *SIAM J. Optim.*, 19(2):655–673, 2008. 10

Index