HIROSHIMA'S THEOREM AND MATRIX NORM INEQUALITIES

MINGHUA LIN AND HENRY WOLKOWICZ

Abstract. In this article, several matrix norm inequalities are proved by making use of the Hiroshima 2003 result on majorization relations.

1. Introduction

In 2003, Hiroshima [7] proved a very beautiful result on majorization relations (Theorem 1.1 below) which has useful applications in quantum physics; see, e.g., [6, 9] and references therein. However, this result seems not widely known in the field of matrix analysis. Indeed, independent of the Hiroshima paper, the authors of the current paper derived the following special case of Theorem 1.1:

$$H = \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \Longrightarrow \|H\| \leq \|A+B\| \tag{1.1}$$

for any unitarily invariant norm; see [13]. (Here $H \geq 0$ denotes positive semidefinite.)

We remark that a sharper observation that entails (1.1) is the following

$$H = \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \Longrightarrow H = \frac{1}{2}\Big(U(A+B)U^* + V(A+B)V^*\Big)$$

for some isometries U, V; see [3] and its extensions in [4].

In this paper we look at several classes of matrix norm inequalities. This includes *commuting type* inequalities and inequalities involving *contractive matrices*.

Before introducing Hiroshima's result, we fix our notation. The set of $m \times n$ complex matrices is denoted by $\mathbb{M}_{m\times n}$ with $\mathbb{M}_n := \mathbb{M}_{n\times n}$. The identity matrix in $\mathbb{M}_n$ is I_n, or I for short if the dimension is clear from the context. Let $A \in \mathbb{M}_{m\times n}$. then A^T, A^* denotes the transpose, conjugate transpose of A, respectively. The absolute value of A is given by $|A| = (A^*A)^{1/2}$, i.e., the positive square root of A^*A. We denote the j-th largest singular value of A by $\sigma_j(A)$. Thus $\sigma_j(A) = \lambda_j(|A|) = \sqrt{\lambda_j(A * A)}$, where λ_j denotes the

2010 *Mathematics Subject Classification.* 15A60, 47A30.

Key words and phrases. Hiroshima's theorem, matrix inequalities, commuting type inequalities, unitarily invariant norm.

j-largest eigenvalue. If $A \in \mathbb{M}_n$, then the trace of A is denoted by $\mathrm{tr}A$. For two Hermitian matrices $A, B \in \mathbb{M}_n$, we write $A \geq B$ to mean $A - B$ is positive (semidefinite), so $A \geq 0$ means A is positive. A norm $\|\cdot\|$ on $\mathbb{M}_n$ is called unitarily invariant if $\|UAV\| = \|A\|$ for any $A \in \mathbb{M}_n$ and any unitary matrices $U, V \in \mathbb{M}_n$.

The tensor product $\mathbb{M}_m \otimes \mathbb{M}_n$ is canonically identified with $\mathbb{M}_m(\mathbb{M}_n)$. Here $\mathbb{M}_m(\mathbb{M}_n)$ is the space of $m \times m$ block matrices with entries in $\mathbb{M}_n$. Let $A_j \in \mathbb{M}_m$, $B_j \in \mathbb{M}_n$, $j = 1, \ldots, p$, and consider the tensor product $H = \sum_{j=1}^p A_j \otimes B_j$. As $H \in \mathbb{M}_m(\mathbb{M}_n)$, we can write $H = [H_{i,j}]$ with $H_{i,j} \in \mathbb{M}_n$. The partial trace of H is defined as (see, e.g., [14, p. 31]).

$$\mathrm{tr}_1 H = \sum_{j=1}^{p} (\mathrm{tr}A_j) B_j.$$

It is readily verified that $\mathrm{tr}_1 H = \sum_{j=1}^m H_{j,j}$. The partial transpose (map) $H \mapsto H^\tau$ is defined on $\mathbb{M}_m \otimes \mathbb{M}_n$ as $H^\tau = \sum_{j=1}^p A_j^T \otimes B_j$. If H and H^τ are both positive, then we say H is positive partial transpose.

Hiroshima's result, in our notation, can be stated as follows.

Theorem 1.1. [7, Theorem 1] *Let $H = [H_{i,j}] \in \mathbb{M}_m(\mathbb{M}_n)$. If $I_m \otimes \mathrm{tr}_1 H \geq H \geq 0$, then*

$$\|H\| \leq \|\mathrm{tr}_1 H\| = \left\| \sum_{j=1}^{m} H_{j,j} \right\|, \tag{1.2}$$

for any unitarily invariant norm. In particular, if H is positive partial transpose, then (1.2) holds.

We remark that the observation

$$H \text{ is positive partial transpose} \Longrightarrow I_m \otimes \mathrm{tr}_1 H \geq H$$

is due to Horodecki et al.[8].

In this paper, we make use of Theorem 1.1 to prove some new results on matrix norm inequalities. It is expected that Hiroshima's theorem will become a practical tool in the field of matrix analysis.

2. Norm inequalities of "commuting" type

Our first result is the following norm inequality of *commuting type*. It is an immediate consequence of Hiroshima's result.

Proposition 2.1. *Let $X_1, \dots, X_k \in \mathbb{M}_{m\times n}$ such that $X_i^* X_j$ is Hermitian for all $1 \le i, j \le k$. Then*

$$\left\| \sum_{j=1}^{k} X_j X_j^* \right\| \le \left\| \sum_{j=1}^{k} X_j^* X_j \right\| \tag{2.1}$$

for any unitarily invariant norm.

Proof. Denote $H = [H_{i,j}] := \begin{bmatrix} X_1^* \\ \vdots \\ X_k^* \end{bmatrix} \begin{bmatrix} X_1 & \cdots & X_k \end{bmatrix}$. Then $H \in \mathbb{M}_k(\mathbb{M}_n)$ is positive and $H_{i,j} = H_{j,i}$; therefore H equals its partial transpose. Thus by Theorem 1.1, we have

$$\|H\| \le \left\| \sum_{j=1}^{k} H_{jj} \right\| = \left\| \sum_{j=1}^{k} X_j^* X_j \right\|.$$

On the other hand,

$$\begin{aligned} \|H\| &= \left\| \begin{bmatrix} X_1^* \\ \vdots \\ X_k^* \end{bmatrix} \begin{bmatrix} X_1 & \cdots & X_k \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} X_1 & \cdots & X_k \end{bmatrix} \begin{bmatrix} X_1^* \\ \vdots \\ X_k^* \end{bmatrix} \right\| \\ &= \left\| \sum_{j=1}^{k} X_j X_j^* \right\|. \end{aligned}$$

The desired result follows. □

The special case of Proposition 2.1 when $k = 2$ has been observed in [13, Corollary 2.2].

The inequality (2.1) is elegant as an inequality of *commuting type*. To the authors' best knowledge, another example of commuting type norm inequality is the following, and there are no others.

Proposition 2.2. [2, p. 254] *Let $X, Y \in \mathbb{M}_{m\times n}$ such that X^*Y is Hermitian. Then*

$$\|X^*Y + Y^*X\| \le \|XY^* + YX^*\| \tag{2.2}$$

for any unitarily invariant norm.

The usefulness of inequality (2.2) has been demonstrated in matrix analysis, see e.g., [2, p. 263], [10] and [15, p. 67]. We now present an application of Proposition 2.1 following the spirit of [2, p. 263].

Theorem 2.3. *Let $A, B, X \in \mathbb{M}_n$. Then*

$$\|(AA^* + XB^*BX^*) \oplus (B^*B + X^*AA^*X)\| \\ \leq \|(A^*A + A^*XX^*A) \oplus (BB^* + BX^*XB^*)\| \tag{2.3}$$

for any unitarily invariant norm.

Proof. We first prove the case where A, B are Hermitian. Consider

$$T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \ S = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}.$$

Take $X_1 = T$, $X_2 = ST$, as $X_1^* X_2 = TST$ is Hermitian. Then by Proposition 2.1 for $k = 2$, we have (see also [4, Corollary 2.5])

$$\|T^2 + ST^2S\| \leq \|T^2 + TS^2T\|,$$

i.e.,

$$\|(A^2 + XB^2X^*) \oplus (B^2 + X^*A^2X)\| \\ \leq \|(A^2 + AXX^*A) \oplus (B^2 + BX^*XB)\|. \tag{2.4}$$

For the general case, consider the polar decompositions $A = |A|U$, $B = |B|V$. Then (2.4) yields

$$\|(|A|^2 + Y|B|^2Y^*) \oplus (|B|^2 + Y^*|A|^2Y)\| \\ \leq \|(|A|^2 + |A|YY^*|A|) \oplus (|B|^2 + |B|Y^*Y|B|)\|. \tag{2.5}$$

for any $Y \in \mathbb{M}_n$.

As $|A| = AU^* = UA^*$, $|B| = BV^* = VB^*$, we have $|A|^2 = AA^* = UA^*AU^*$, $|B|^2 = BB^* = VB^*BV^*$. Substituting these into (2.5) and setting $Y = XV^*$ gives (2.3). □

Remark 2.4. In particular, for the Schatten-p norm, (2.3) leads to

$$\|AA^* + XB^*BX^*\|_p^p + \|B^*B + X^*AA^*X\|_p^p \\ \leq \|A^*A + A^*XX^*A\|_p^p + \|BB^* + BX^*XB^*)\|_p^p.$$

For the trace norm, (2.3) becomes an equality, and so we have the following determinantal inequality

$$\det(AA^* + XB^*BX^*) \det(B^*B + X^*AA^*X) \\ \geq \det(A^*A + A^*XX^*A) \det(BB^* + BX^*XB^*).$$

3. Norm inequalities involving contractions

In this and the subsequent section, we are mainly concerned with the application of Theorem 1.1 when $H \in \mathbb{M}_2(\mathbb{M}_n)$. We restate it here as a lemma.

Lemma 3.1. *Let* $H = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ *be positive. Furthermore, if* H *is positive partial transpose, i.e.,* $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}$ *is positive, then*

$$\|H\| \le \|A + B\|$$

for any unitarily invariant norm.

The Hua matrix, e.g., [17], is given by

$$\mathbf{H} := \begin{bmatrix} (I - A^*A)^{-1} & (I - B^*A)^{-1} \\ (I - A^*B)^{-1} & (I - B^*B)^{-1} \end{bmatrix},$$

where $A, B \in \mathbb{M}_{m\times n}$ are strictly contractive. Recall that $X \in \mathbb{M}_{m\times n}$ is called strictly contractive if $I - X^*X$ is positive and nonsingular.

A fundamental fact for the Hua matrix is that it is positive. This follows from the Schur complement and an elegant matrix identity, which also carries the name of Hua (e.g., [17]):

$$(I-B^*B)-(I-B^*A)(I-A^*A)^{-1}(I-A^*B) = -(A-B)^*(I-AA^*)^{-1}(A-B)$$

with A, B strictly contractive.

But it is only recently observed that $\mathbf{H}$ is positive partial transpose; see [1, 17]. An interesting application of this observation can be found in [12].

The next lemma plays an important role in our analysis.

Lemma 3.2. [16, Theorem 1] *Let* $H = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{m+n}$ *be positive with* $A \in \mathbb{M}_m$, $B \in \mathbb{M}_n$. *Then*

$$2\sigma_j(X) \le \sigma_j(H), \ \ j = 1, \dots, \min\{m, n\}.$$

It should be mentioned that Lemma 3.2 has several variants. We refer to [16] and references therein for equivalent forms.

Fan's dominance theorem (e.g., [2, p. 93]) reveals an important relation between singular value inequalities and norm inequalities. More precisely, let $A, B \in \mathbb{M}_n$. Then the following statements are equivalent:

(i) $\sum_{j=1}^k \sigma_j(A) \ge \sum_{j=1}^k \sigma_j(B)$, for $k = 1, \dots, n$;
(ii) $\|A\| \ge \|B\|$ for any unitarily invariant norm $\|\cdot\|$.

Thus Lemma 3.2 is strong enough to entail

$$H = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \ge 0 \Longrightarrow 2\|X\| \le \|H\| \tag{3.1}$$

for any unitarily invariant norm.

The main result of this section states a relation between the norm of diagonal blocks of $\mathbf{H}$ and the norm of its off diagonal block.

Theorem 3.3. *Let $A, B \in \mathbb{M}_{m\times n}$ be strictly contractive. Then*

$$2\|(I - A^*B)^{-1}\| \le \|(I - A^*A)^{-1} + (I - B^*B)^{-1}\| \tag{3.2}$$

for any unitarily invariant norm.

Proof. As $\mathbf{H}$ is positive partial transpose, (3.2) follows immediately from Lemma 3.1 and (3.1). □

From the proof of Theorem 3.3, we find that in proving certain norm inequalities, it suffices to show the corresponding (block) matrix is positive partial transpose.

One may suspect that Theorem 3.3 is a special case of a much more general result. In particular, it is tempted to ask whether it is true

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} \ge 0 \Longrightarrow 2\|X\| \le \|A + B\|? \tag{3.3}$$

The answer is no, as the following example shows.

Example 3.4. Take

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \; X = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

It is easy to check in this case $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is positive. A simple calculation gives $\sigma_1(X) = 2$, $\sigma_2(X) = 0$ and $\sigma_1(A + B) = \sigma_2(A + B) = 2$. Thus in this case, by Fan's dominance theorem, $2\|X\| \ge \|A + B\|$ holds for every unitarily invariant norm.

Remark 3.5. In spite of the failure of (3.3), we have the following: Assume $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ is positive. Then

$$\|X + X^*\| \le \|A + B\| \tag{3.4}$$

for any unitarily invariant norm. Indeed,

$$\begin{aligned} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \ge 0 \;\; &\Longrightarrow \;\; \begin{bmatrix} A + B & X + X^* \\ X + X^* & A + B \end{bmatrix} \ge 0 \\ &\Longrightarrow \prod_{j=1}^{k} \sigma_j(X + X^*) \le \prod_{j=1}^{k} \sigma_j(A + B), \qquad k = 1, \dots, n, \end{aligned}$$

which is stronger than (3.4).

4. Miscellaneous

Proposition 4.1. *Let $A, B \in \mathbb{M}_n$ be positive and $U \in \mathbb{M}_n$ be unitary. Then*

$$\|A + UB\| \le \|A + B + UBU^*\| \tag{4.1}$$

for any unitarily invariant norm.

Proof. First, we observe that

$$\begin{aligned} H &:= \begin{bmatrix} A + B + UBU^* & A + UB \\ A + BU^* & A + B + UBU^* \end{bmatrix} \\ &= \begin{bmatrix} A & A \\ A & A \end{bmatrix} + \begin{bmatrix} UBU^* & UB \\ BU^* & B \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & UBU^* \end{bmatrix} \ge 0. \end{aligned}$$

Second,

$$\begin{aligned} H^\tau &= \begin{bmatrix} A + B + UBU^* & A + BU^* \\ A + UB & A + B + UBU^* \end{bmatrix} \\ &= \begin{bmatrix} A & A \\ A & A \end{bmatrix} + \begin{bmatrix} B & BU^* \\ UB & UBU^* \end{bmatrix} + \begin{bmatrix} UBU^* & 0 \\ 0 & B \end{bmatrix} \ge 0. \end{aligned}$$

Thus, H is positive partial transpose. The desired result then follows by Lemma 3.1 and (3.1). □

On the other hand, Lee [11] proved the following.

Proposition 4.2. *Let $A, B \in \mathbb{M}_n$. Then*

$$\|A + B\| \le \sqrt{2}\||A| + |B|\|$$

for any unitarily invariant norm. Equivalently, if $A, B \ge 0$, then for any unitary matrix $U \in \mathbb{M}_n$,

$$\|A + UB\| \le \sqrt{2}\|A + B\|. \tag{4.2}$$

There is no obvious relation between (4.1) and (4.2). However, when $A \ge 0$ and $B = I$, we have a stronger inequality

$$\|A + U\| \le \|A + I\|,$$

which is due to Fan and Hoffman [5, Theorem 1'].

Acknowledgements. The authors wish to thank the referee for their insightful comments which greatly improved the presentation.

References

[1] T. Ando, Positivity of operator-matrices of Hua-type, Banach J. Math. Anal. 2 (2008) 1-8.

[2] R. Bhatia, Matrix Analysis, GTM 169, Springer-Verlag, New York, 1997.

[3] J.-C. Bourin, E.-Y. Lee, Decomposition and partial trace of positive matrices with Hermitian blocks, Internat. J. Math. 24 (2013) 1350010.

[4] J.-C. Bourin, E.-Y. Lee and M. Lin, Positive matrices partitioned into a small number of Hermitian blocks, Linear Algebra Appl. 438 (2013) 2591-2598. classes, J. London Math. Soc. 37 (1988) 148-157.

[5] K. Fan, A. Hoffman, Some metric inequalities in the space of matrices, Proc. Amer. Math. Soc. 6 (1955) 111-116.

[6] O. Gühne, G. Tóth, Entanglement detection, Phys. Rep. 474 (2009) 1-75.

[7] T. Hiroshima, Majorization criterion for distillability of a bipartite quantum state, Phys. Rev. Lett. 91 (2003) no. 5 057902, 4 pp.

[8] M. Horodecki, P. Horodecki and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, Phys. Lett. A 223 (1996) 1-8.

[9] R. Horodecki, et al. Quantum entanglement, Rev. Mod. Phys. 81 (2009) 865-942.

[10] F. Kittaneh, Norm inequalities for sums of positive operators, J. Oper. Theory 48 (2002) 95-103.

[11] E.-Y. Lee, How to compare the absolute values of operator sums and the sums of absolute values, 6 (2012) 613-619.

[12] M. Lin, A Lewent type determinantal inequality, Taiwanese J. Math. 17 (2013) 1303-1309.

[13] M. Lin, H. Wolkowicz, An eigenvalue majorization inequality for positive semidefinite block matrices, Linear Multilinear Algebra 60 (2012) 1365-1368.

[14] D. Petz, Matrix Analysis with some Applications, 2011. Available at http://www.math.bme.hu/˜petz

[15] B. Simon, Trace Ideals and Their Applications, Second ed., Amer. Math. Soc., Providence, RI, 2005.

[16] Y. Tao, More results on singular value inequalities of matrices, Linear Algebra Appl. 416 (2006) 724-729.

[17] G. Xu, C. Xu and F. Zhang, Contractive matrices of Hua type, Linear Multilinear Algebra 59 (2011) 159-172.

Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada V8W 3R4

E-mail address: linm@uvic.ca; mlin87@ymail.com

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON, Canada N2L 3G1.

E-mail address: hwolkowicz@uwaterloo.ca