Extensions of Samuelson's Inequality

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In this note we obtain upper and lower bounds for the kth largest number in a set of real numbers in terms of their mean and standard deviation. For each inequality necessary and sufficient conditions for equality are given.

KEY WORDS: Bounds for the kth largest; Cauchy-Schwarz inequality; Standard deviation; Order statistics; Inequalities; Subsample mean.

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1. THE INEQUALITIES

Samuelson (1968) and Arnold (1974) have shown that for any real numbers $x_1 \geq x_2 \geq \ldots \geq x_n$ the following inequalities hold:

$$x_1 \leq \bar{x} + s(n-1)^{1/2}, \qquad (1.1)$$

$$x_n \geq \bar{x} - s(n-1)^{1/2}, \qquad (1.2)$$

where the mean $\bar{x} = \sum x_i/n$ and the variance $s^2 = \sum (x_i - \bar{x})^2/n$. In this note we show that x_1 and x_n must also satisfy

$$x_1 \geq \bar{x} + s/(n-1)^{1/2}, \quad (1.3)$$

$$x_n \leq \bar{x} - s/(n-1)^{1/2}. \quad (1.4)$$

Furthermore, for $k = 1, 2, \ldots, n$,

$$\bar{x} - s\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq x_k \leq \bar{x} + s\left(\frac{n-k}{k}\right)^{1/2}. \quad (1.5)$$

When $k = 1$, the right side of (1.5) becomes (1.1), while when $k = n$, the left side becomes (1.2).

Equality holds in (1.1) if and only if equality holds in (1.4) if and only if $x_2 = x_3 = \ldots = x_n$. Equality holds in (1.2) if and only if equality holds in (1.3) if and only if $x_1 = x_2 = \ldots = x_{n-1}$. Equality holds on the left side of (1.5) if and only if

$$x_1 = x_2 = \ldots = x_{k-1} \text{ and } x_k = x_{k+1} = \ldots = x_n, \quad (1.6)$$

while equality holds on the right side of (1.5) if and only if

$$x_1 = x_2 = \ldots = x_k \text{ and } x_{k+1} = x_{k+2} = \ldots = x_n. \quad (1.7)$$

2. PROOFS

To prove inequalities (1.3) and (1.4) we note that

$$n^2(x_1 - \bar{x})^2 = [\sum (x_1 - x_j)]^2 = \sum (x_1 - x_j)^2 + \sum_{j \neq k} (x_1 - x_j)(x_1 - x_k)$$
$$\geq \sum (x_1 - x_j)^2 = \sum (x_1 - \bar{x} + \bar{x} - x_j)^2 = n[(x_1 - \bar{x})^2 + s^2], \quad (2.1)$$

from which (1.3) follows at once. Expanding $n^2(\bar{x} - x_n)^2$ in a similar way yields (1.4).

To prove (1.5) we use a form of the Cauchy-Schwarz inequality

$$|\mathbf{w}'\mathbf{C}\mathbf{x}| \leq (\mathbf{w}'\mathbf{C}\mathbf{w} \cdot \mathbf{x}'\mathbf{C}\mathbf{x})^{1/2} = s(n\mathbf{w}'\mathbf{C}\mathbf{w})^{1/2}, \quad (2.2)$$

where $\mathbf{x} = \{x_i\}$ and $\mathbf{w}$ are $n \times 1$ vectors and $\mathbf{C}$ is the $n \times n$ idempotent "centering" matrix $\mathbf{I} - \mathbf{e}\mathbf{e}'/n$, with $\mathbf{e} = (1, 1, \ldots, 1)'$. Putting $\mathbf{w} = \sum_{j=k}^{l} \mathbf{e}_j/(l - k + 1)$ gives $\mathbf{w}'\mathbf{C}\mathbf{x} = \bar{x}_{(k,l)} - \bar{x}$, where $\mathbf{e}_j$ is the jth column of $\mathbf{I}_n$ and $\bar{x}_{(k,l)}$ is the "subsample mean" $\sum_{j=k}^{l} x_j/(l - k + 1)$. Moreover, $\mathbf{w}'\mathbf{C}\mathbf{w} = (l - k + 1)^{-1} - n^{-1}$. Hence (2.2) implies

$$\bar{x} - s\left(\frac{k-1}{n-k+1}\right)^{1/2} \leq \bar{x}_{(k,n)} \leq \bar{x}_{(k,l)}$$
$$\leq \bar{x}_{(1,l)} \leq \bar{x} + s\left(\frac{n-l}{l}\right)^{1/2}, \quad (2.3)$$

which reduces to (1.5) when $l = k$.

The conditions for equality in (1.1) through (1.5) follow directly from the conditions for equality in (2.1), (2.2), and (2.3), noting that equality holds in (2.2) if and only if $\mathbf{x} = a\mathbf{w} + b\mathbf{e}$, for some scalars a and b.

3. NUMERICAL EXAMPLES

Samuelson (1968, p. 1522) noted that "to assert that a man is a million standard deviations above the mean is to assert a falsehood (there being less than 4 billion men extant)." The *Guinness Book of World Records* (1978 ed., p. 171) estimates that in 1977 the world population was 4,126 million. Therefore, the inequality (1.1) shows that in 1977 a person could not be more than 64,234 standard deviations (s.d.'s) above the mean. Furthermore, the inequality (1.5) shows that the 2nd largest person cannot be more than 45,421 s.d.'s above the mean, and so on. Moreover, if n is somewhat smaller, for example, 17, then the largest person cannot be more than 4 s.d.'s or less than ¼ s.d. above the mean, while the 2nd largest person cannot be more than 2¾ s.d.'s above or less than ¼ s.d. below the mean.

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