SHADOW PRICES FOR AN UNSTABLE CONVEX PROGRAM

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ABSTRACT. We find shadow prices or marginal values for the convex program

$$\text{minimize } f(x) \text{ subject to } g^k(x) \le 0,\ k \in P = \{1,\dots,m\}$$

when no constraint qualification holds at the optimum point. Some of these multipliers must then have an 'infinite' value.

1. *Introduction.*

Consider the convex programming problem

$$(P) \qquad \begin{cases} \text{minimize } f(x) \\ \text{subject to} \\ \qquad g^k(x) \le 0,\ k \in P = \{1,\dots,m\} \\ \qquad x \in R^n , \end{cases}$$

where the functions $f, g^k : R^n \to R$ are convex. (P) may represent the problem of maximizing the profit of an industrial plant where: the variable $x = (x_j)$ represents the amounts of the j products produced; the negative of the objective function f is the dollars of return; and the constraints $g^k = h^k - b_k$ represent the dependence of the 'scarce' k^{th} resource with amount b_k available.

If x solves (P) and a suitable constraint qualification holds at x, then it is well-known that there exists an optimal Kuhn-Tucker multiplier vector $\lambda = (\lambda_k) \in R^m$ such that $\lambda_k \ge 0$, $\lambda_k g^k(x) = 0$, and x is the global minimum of the Lagrangian

$$F(y) = f(y) + \sum_k \lambda_k g^k(y) ,$$

see e.g. [7]. If

$$v(e) = \inf\{f(y) : g(y) \le e\}$$

is the *perturbation function*, see e.g. [7], where $g(y) = (g^k(y))$

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and $e = (e_k)$ are vectors in R^m, then the Kuhn-Tucker multipliers represent sensitivity coefficients or shadow prices and

$$v(e) - v(0) \geq - \sum_k \lambda_k e_k .$$

This means that we can obtain a lower bound on the marginal rate of decrease in the optimal value of the objective function when the amount of the resources available changes, i.e., program (P) is stable. For example, if an additional amount of some k^{th} resource can be purchased for y dollars per unit, then this purchase is not economical if $y > \lambda_k$, i.e. if the marginal return from the optimal allocation of an additional unit of the k^{th} resource is less than its marginal cost.

However, an optimal multiplier vector may not exist, see e.g. [2], [3], [4]. This is equivalent to the fact that the marginal rate of decrease in the optimal value of the objective function with respect to an increase in *all* the resources available is $-\infty$, i.e. program (P) is unstable, see [7]. This means that it is economical to purchase 'small' additional amounts of all the resources, no matter the cost. But, it may be inconvenient or even impossible to purchase additional amounts of all the resources.

The purpose of this paper is to isolate the set of 'vital' resources, i.e. the resources which lead to an infinite marginal increase in the dollars of return. We will see that purchasing additional amounts of just one resource is usually sufficient. We will also obtain finite shadow prices for the 'nonvital' resources.

Section 2 gives several definitions and preliminary results, while Section 3 presents various optimality and regularity criteria for (P). The main result then appears in Section 4.

2. *Preliminaries.*

For the convex program (P), we assume that the *feasible set*

$$F = \{x: g^k(x) \leq 0, \text{ for all } k \in P\}$$

is nonempty. The set of *binding (active) constraints* at $x \in F$ is

$$P(x) = \{k \in P: g^k(x) = 0\} .$$

An important subset of P independent of x is the *equality set*, see e.g. [1],

$$P^= = \{k \in P: g^k(x) = 0, \text{ for all } x \in F\} .$$

We then denote

$$P^<(x) = P(x) \backslash P^= .$$

Following [2], we denote the relations

"relation" is "=", "<", "≤", ">" or "≥"

and set

$$D_g^{\text{"relation"}}(x) = \{d: \text{there exists } \bar{\alpha} > 0 \text{ with } g(x+\alpha d) \text{ "relation" } g(x), \text{ for all } 0 < \alpha \leq \bar{\alpha}\} .$$

For simplicity of notation we let

$$D_k^{\text{"relation"}}(x) = D_{g^k}^{\text{"relation"}}(x)$$

and

$$D_\Omega^{\text{"relation"}}(x) = \bigcap_{k \in \Omega} D_k^{\text{"relation"}}(x), \text{ for } \Omega \subset P .$$

Note that, see e.g. [2],

$$\text{conv } D_\Omega^=(x) \subset D_\Omega^\leq(x) ,$$

where conv denotes convex hull;

(2.1) $$D_{P^=}^=(x) \cap D_{P^<(x)}^<(x) \neq \emptyset ;$$

and, see [5],

(2.2) $$D_{P^=}^=(x) = D_{P^=}^\leq(x) \quad \text{(and so is convex)} .$$

For a convex function $g: R^n \to R \cup \{\infty\}$, the *directional derivative* of g at x in the direction d is defined as

$$\nabla g(x;d) = \lim_{t \downarrow 0} \frac{g(x+td) - g(x)}{t} .$$

If $g(x)$ is finite, then the directional derivative exists in all directions d, although it may be plus or minus infinity. A vector $\phi \in R^n$ is said to be a *subgradient* of g at x if

$$g(z) \geq g(x) + \phi(z-x), \quad \text{for all } z \in R^n .$$

The set of all subgradients of g at x is then called the *subdifferential* of g at x and is denoted $\partial g(x)$. If g is differentiable at x, then

$$\partial g(x) = \{\nabla g(x)\}; \quad \nabla g(x;d) = \nabla g(x)d ,$$

where $\nabla g(x)$ is the *gradient* of g at x and, for two vectors u and v in R^n, uv denotes the dot product. In general, if x is in the interior of $\mathrm{dom}(g)$ (the domain of g, i.e. the set of points where g is finite), then $\partial g(x)$ is a nonempty compact convex set and

(2.3) $$\nabla g(x;d) = \max\{\phi d : \phi \in \partial g(x)\} .$$

If M is a set in R^n, then the *nonnegative polar* of M is

$$M^+ = \{\phi \in R^n : \phi m \geq 0, \quad \text{for all } m \in M\} .$$

Note that

(2.4) $$(M \cap N)^+ = \overline{M^+ + N^+} ; \quad C^{++} = \overline{\text{cone}}\ C ,$$

where M and N are closed convex cones, $\overline{\cdot}$ denotes closure while cone C is the convex cone generated by C. Some useful properties of a convex function $h: R^n \to R$ are:

(2.5) $$D_h^<(x) \cup D_h^=(x) \cup D_h^>(x) = R^n ;$$

(2.6) $$\partial h(x)^+ \subset D_h^\geq(x) ;$$

(2.7) $$D_h^\geq(x) = \{d : \nabla h(x;d) \geq 0\} ;$$

(2.8) $$D_h^<(x) = \{d : \nabla h(x;d) < 0\} ;$$

(2.9) $$D_h^<(x)^+ = -\text{cone}\ \partial h(x) , \quad \text{when } 0 \notin \partial h(x) .$$

For these and other related results, see e.g. [2], [3], [9].

For every subset Ω of $P(x)$, the *linearizing cone* at $x \in F$ with respect to Ω is

(2.10) $$C_\Omega(x) = \{d : \phi d \leq 0, \quad \text{for all } \phi \in \partial g^k(x) \text{ and all } k \in \Omega\}.$$

By (2.3), we see that

(2.11) $$C_\Omega(x) = \{d : \nabla g^k(x;d) \leq 0, \quad \text{for all } k \in \Omega\} .$$

The *cone of subgradients* at x is

(2.12) $$B_\Omega(x) = \left\{\phi : \phi = \sum_{k \in \Omega} \lambda_k \phi^k , \quad \text{for some } \lambda_k \geq 0 \text{ and } \phi^k \in \partial g^k(x)\right\} .$$

The linearizing cone and the cone of subgradients have the following dual property, see e.g. [13],

(2.13) $$\overline{B_\Omega(x)} = -C_\Omega^+(x) .$$

Following [13], we introduce the set of *'badly behaved' constraints* at $x \in F$,

(2.14) $$P^b(x) = \{k \in P^= : D_k^>(x) \cap C_{P(x)}(x) \setminus \overline{D^=_{P^=}(x)} \neq \emptyset\} .$$

These are the constraints that create problems in the Kuhn-Tucker theory. It was shown in [13] that

$$P^b(x) = \emptyset \text{ and } B_{P(x)}(x) \text{ is closed}$$

is a *weakest constraint qualification* at x. We now let

(2.15) $$P_f^b(x) = \{k \in P^= : D_f^<(x) \cap D_k^>(x) \cap C_{P(x)}(x) \neq \emptyset\} .$$

We will see that $P_f^b(x)$ corresponds exactly to the set of 'vital' resources at x if x solves (P), i.e. an increment in the k^{th} resource, $k \in P_f^b(x)$, leads to an infinite marginal decrease in the optimal value of the objective function.

3. *Optimality Conditions.*

In this section we present several optimality conditions which hold under different constraint qualifications as well as some which hold without any constraint qualification. We also present a characterization of regularity of (P) and find a 'bad' direction in the case that (P) is not regular.

Recall that: (i) Slater's condition holds if there exists $\hat{x}$ such that

(3.1) $$g^k(\hat{x}) < 0, \quad \text{for all } k \in P ,$$

and (ii) x is a Kuhn-Tucker point if

(3.2) $$\partial f(x) \cap -B_{P(x)}(x) \neq \emptyset .$$

We call program (P) *regular* if (3.2) characterizes optimality of x. Note that (3.2) is alwyas sufficient for optimality but necessity may fail in the absence of a suitable constraint qualification, see e.g. [2], [3], [4].

PROPOSITION 3.1. (e.g. Zoutendijk [14]). *Suppose that Slater's condition holds for* (P). *Then* $x \in F$ *solves* (P) *if and only if*

$$D_f^<(x) \cap D_{P(x)}^<(x) = \emptyset .$$

PROPOSITION 3.2. *Suppose that* (P) *is regular, i.e. the optimal point is a Kuhn-Tucker point. Then* $x \in F$ *solves* (P) *if and only if*

(3.3) $$D_f^<(x) \cap C_{P(x)}(x) = \emptyset .$$

Proof. If $x \in F$ is not optimal, then there exists $y \in F$ such that $f(y) < f(x)$. This implies that

$$y - x \in D_f^<(x) \cap C_{P(x)}(x) .$$

Conversely, if $x \in F$ is optimal, then by hypothesis there exists

$$\phi \in \partial f(x) \cap -B_{P(x)}(x) .$$

Therefore, by (2.13), (2.3), (2.4), and (2.7), we get that

$$C_{P(x)}(x) = -B_{P(x)}(x)^+ \subset \{\phi\}^+ \subset D_f^{\geq}(x) .$$

□

PROPOSITION 3.3. ([1], [2], [3]). $x \in F$ *solves* (P) *if and only if*

(3.4) $$D_f^<(x) \cap D^<_{P^<(x)}(x) \cap D^=_{P^=}(x) = \emptyset$$

if and only if

$$\partial f(x) \cap (-B_{P^<(x)}(x) + D^=_{P^=}(x)^+) \neq \emptyset .$$

COROLLARY 3.1. ([13]). $x \in F$ *solves* (P) *if and only if*

(3.5) $$\partial f(x) \cap (-B_{P(x)}(x) + D^=_{P^=}(x)^+) \neq \emptyset .$$

COROLLARY 3.2. $x \in F$ *solves* (P) *if and only if*

(3.6) $$D_f^<(x) \cap C_{P(x)}(x) \cap D^=_{P^=}(x) = \emptyset .$$

Proof. Since $C_{P(x)}(x) = C_{P^<(x)}(x) \cap C_{P^=}(x)$ and $D^=_{P^=}(x) \subset C_{P^=}(x)$, we get that

$$D_f^<(x) \cap C_{P(x)}(x) \cap D^=_{P^=}(x) = D_f^<(x) \cap C_{P^<(x)}(x) \cap D^=_{P^=}(x) .$$

Suppose that (3.6) fails. Then there exists

$$d \in D_f^<(x) \cap C_{P^<(x)}(x) \cap D^=_{P^=}(x) .$$

By (2.1) we can find

(3.7) $$\hat{d} \in D^<_{P^<(x)}(x) \cap D^=_{P^=}(x) .$$

Let

$$d_\lambda = \lambda \hat{d} + (1-\lambda) d , \quad \text{for } 0 \leq \lambda \leq 1 .$$

Now, since $C_{P^<(x)}(x)$ is convex, $\hat{d} \in D^<_{P^<(x)}(x) \subset \operatorname{int} C_{P^<(x)}(x)$ and (see (2.2)) $D^=_{P^=}$ is convex, we conclude that

$$d_\lambda \in D^<_f(x) \cap D^<_{P^<}(x) \cap D^=_{P^=}(x) ,$$

for $\lambda > 0$ sufficiently small. By the above proposition, x is not optimal. This argument is reversable. □

Recall that

$$(3.8) \qquad P^b_f(x) = \{k \in P^= : D^<_f(x) \cap D^>_k(x) \cap C_{P(x)}(x) \neq \emptyset\} .$$

Let the set Ω satisfy

$$(3.9) \qquad P^b_f(x) \subset \Omega \subset P^= .$$

We will now see that $P^=$ may be replaced by Ω.

PROPOSITION 3.4. $x \in F$ *solves* (P) *if and only if*

$$(3.10) \qquad D^<_f(x) \cap C_{P(x)}(x) \cap D^=_\Omega(x) = \emptyset .$$

Proof. It is sufficient to show that (3.10) is equivalent to (3.6). That (3.10) implies (3.6) is clear. To prove the converse, suppose that

$$d \in D^<_f(x) \cap C_{P(x)}(x) \cap D^=_\Omega(x) \backslash D^=_{P^=}(x) .$$

By (3.8) and (3.9), we get that

$$d \in D^<_f(x) \cap C_{P(x)}(x) \cap D^\leq_{P^=}(x) .$$

But now (2.2) yields a contradiction. □

COROLLARY 3.3. $x \in F$ *solves* (P) *if and only if*

$$D^<_f(x) \cap C_{P(x)}(x) \cap D^\leq_\Omega(x) = \emptyset .$$

Proof. The proof is identical to the one above. Note that this argument also shows that

$$C_{P(x)}(x) \cap D^=_\Omega(x) = C_{P(x)}(x) \cap D^\leq_{\Omega(x)}$$

is a convex set. □

COROLLARY 3.4. *Suppose that* $P^b_f(x) = \emptyset$. *Then* $x \in F$ *solves* (P) *if and only if*

$$D^<_f(x) \cap C_{P(x)}(x) = \emptyset .$$

COROLLARY 3.5. [13]. *Suppose that both* $\operatorname{conv} D^=_\Omega(x)$ *and* $-B_{P(x)}(x) + D^=_\Omega(x)^+$ *are closed. Then* $x \in F$ *solves* (P) *if and only if*

$$(3.11) \qquad \partial f(x) \cap (-B_{P(x)}(x) + D^=_\Omega(x)^+) \neq \emptyset .$$

Proof. By the Dubovitskii-Milyutin Theorem [6], (3.10) holds if and only if there exists $y_1 \in D^<_f(x)^+$ *and* $y_2 \in (C_{P(x)}(x) \cap D^=_\Omega(x))^+$, such that

$$(3.12) \qquad y_1 + y_2 = 0 , \text{ not both } 0 .$$

Now if $0 \in \partial f(x)$, then x is a minimum for f over R^n, x solves (P), and (3.11) clearly holds. Otherwise, if $0 \notin \partial f(x)$, then by (2.9)

$$D^<_f(x)^+ = -\operatorname{cone} \partial f(x) .$$

Furthermore, by (2.4) and (2.13) and the hypothesis,

$$(C_{P(x)}(x) \cap D^=_\Omega(x))^+ = -B_{P(x)}(x) + D^=_\Omega(x)^+ .$$

The result now follows from (3.12). □

The following result characterizes regularity of (P) at an optimum.

PROPOSITION 3.5. *Suppose that* $B_{P(x)}(x)$ *is closed and* $x \in F$ *solves* (P). *Then the following are equivalent.*

(3.13) $\qquad x$ is a Kuhn-Tucker Point;

$$(3.14) \qquad C_{P(x)}(x) \subset D^\geq_f(x) ;$$

$$P_f^b(x) = \emptyset \, . \tag{3.15}$$

Proof. That (3.13) implies (3.14) follows from Proposition 3.2. The converse follows by (2.6) and since

$$D_f^{\geq}(x)^+ \subset C_{P(x)}^+(x) = -B_{P(x)}(x) \, ,$$

by (2.13). That (3.14) implies (3.15) follows from the definition of $P_f^b(x)$. It remains to show that (3.15) implies (3.14). Suppose that (3.15) holds, but (3.14) fails. But then, by (3.8) and (2.2),

$$\emptyset \neq C_{P(x)}(x) \cap D_f^<(x) \subset D_{P^=}^=(x) \, .$$

This contradicts Proposition 3.4. □

Note that by Proposition 3.2 and the definition of $P_f^b(x)$, we get that

$$P_f^b(x) \neq \emptyset \Rightarrow (P) \text{ is not regular.}$$

However the converse is not necessarily true (when $B_{P(x)}(x)$ is not closed).

Example 3.1.† Consider the program

$$\begin{cases} \min f(y) = y_3 \\ \text{subject to} \\ g^1(y) = \sup\{\phi y: \phi\in K\} \\ y\in R^3 \, , \end{cases}$$

where

$$K = \{\phi = (\phi_i)\in R^3: \phi_1 = 0 \text{ and } (\phi_2-1)^2 + \phi_3^2 - 1 \leq 0\} \, .$$

Then

$$x = (0,0,0)$$

solves (P), $P_f^b(0) = \emptyset$ but (P) is not regular. Note that $B_{P(0)}(0)$ = cone K is not closed.

The following rather technical result shows that when $P_f^b(x) \neq \emptyset$, then we can find a 'bad' direction. This direction will yield the infinite marginal rate of change of the optimal value of (P) with

† This example is due to Prof. J. M. Borwein.

respect to an increase in the right-hand side of the k^{th} constraint, $k\in P_f^b(x)$.

PROPOSITION 3.6. *Suppose that* $x\in F$ *solves* (P) *and* $P_f^b(x) \neq \emptyset$. *Then*

$$\bigcup_{k\in P_f^b(x)} D_f^<(x) \cap D_k^>(x) \cap C_{P(x)}(x) \cap D_{P^<(x)}^{\leq}(x) \neq \emptyset \, . \tag{3.16}$$

Proof. Suppose that (3.16) fails. Then

$$\bigcup_{k\in P_f^b(x)} D_f^<(x) \cap D_k^>(x) \cap C_{P(x)}(x) \subset \bigcup_{j\in P^<(x)} D_j^>(x) \, . \tag{3.17}$$

We will now show that $P_f^b(x) = \emptyset$ and thus contradict the hypothesis. We will do this by constructing a program $(\underset{\sim}{P})$ for which x is optimal and which has the same differential properties as (P).

Let

$$h^k(y) = \sup\{\phi(y-x): \phi\in\partial g^k(x)\} \, . \tag{3.18}$$

Then

$$\partial h^k(x) = \partial g^k(x) \, . \tag{3.18a}$$

For, suppose $\phi\notin\partial g^k(x)$. Since $\partial g^k(x)$ is convex and compact, the hyperplane separation theorem implies that we can find $d\in R^n$ such that

$$\phi d > \sup\{\psi d: \psi\in\partial g^k(x)\} \, .$$

Let $y = d+x$. Then

$$\begin{aligned} \phi(y-x) &= \phi d \\ &> \sup\{\psi d: \psi\in\partial g^k(x)\} \\ &= \sup\{\psi(y-x): \psi\in\partial g^k(x)\} \\ &= h^k(y) - h^k(x), \quad \text{since } h^k(x) = 0 \, . \end{aligned}$$

This implies that $\phi\notin\partial h^k(x)$. Conversely, suppose that $\phi\in\partial g^k(x)$. Then, for all $y\in R^n$,

$$\phi(y-x) \le \sup\{\psi(y-x): \psi\in\partial g^k(x)\}$$
$$= h^k(y) - h^k(x) \ .$$

And thus $\phi\in\partial h^k(x)$. This proves (3.18a). Now consider the convex program

$$(\underset{\sim}{P}) \begin{cases} \min f(y) \\ \text{subject to} \\ g^k(y) \le 0, \ k\in P(x)\setminus P_f^b(x) \\ h^k(y) \le 0, \ k\in P_f^b(x) \ . \end{cases}$$

First, let us show that

(3.19) $\quad x$ is optimal for $(\underset{\sim}{P})$.

Suppose that $y\in F$, the feasible set of (P). Then

$$g^k(y) \le 0, \quad \text{for all} \quad k\in P(x)\setminus P_f^b(x) \ ,$$

and

$$0 = g^k(x) = g^k(y), \quad \text{for all} \quad k\in P_f^b(x) \subset P^= \ .$$

Thus, for all $k\in P_f^b(x)$,

$$0 = \nabla g^k(x;y-x)$$
$$= \sup\{\phi(y-x): \phi\in\partial g^k(x)\}$$
$$= h^k(y) \ .$$

Therefore $y\in\underset{\sim}{F}$, i.e.,

(3.20) $\quad F \subset \underset{\sim}{F}$,

where $\underset{\sim}{F}$ denotes the feasible set of $(\underset{\sim}{P})$. (We let $\sim$ refer to problem $(\underset{\sim}{P})$, e.g. $\underset{\sim}{P}^=$ denotes the equality set of $(\underset{\sim}{P})$.) If $v = f(x)$ is the optimal value of (P), then (3.20) implies that

(3.21) $\quad \underset{\sim}{v} \le v$.

By convexity of the feasible sets and the objective function f, we see that to prove (3.19) it is sufficient to show that equality holds in

(3.21). Suppose not. Then there exists $\underset{\sim}{x} \in \underset{\sim}{F}\setminus F$ such that $f(\underset{\sim}{x}) < f(x)$. Let

$$\underset{\sim}{d} = \underset{\sim}{x} - x \ .$$

Then

$$\underset{\sim}{d} \in D_f^<(x) \ ,$$
$$\nabla g^k(x;\underset{\sim}{d}) \le 0, \quad \text{for all} \quad k\in P(x)\setminus P_f^b(x) \ ,$$

and

$$\nabla h^k(x;\underset{\sim}{d}) \le 0, \quad \text{for all} \quad k\in P_f^b(x) \ .$$

Thus, by (3.18a) we have that

$$C_{\underset{\sim}{P}(x)}(x) = C_{P(x)}(x)$$

and so

$$\underset{\sim}{d} \in D_{f(x)}^< \cap C_{P(x)}(x) \ .$$

Now Corollary 3.3 implies that

$$\underset{\sim}{d} \in D_f^<(x) \cap C_{P(x)}(x) \cap D_k^>(x), \text{ for some } k\in P_f^b(x) \ .$$

But by (3.17) there then exists $j\in P^<(x)$ such that

$$\underset{\sim}{d}\in D_j^>(x) \ .$$

This contradicts the feasibility of $\underset{\sim}{x}$ and proves that equality holds in (3.21). Therefore x is optimal for $(\underset{\sim}{P})$.

By (3.18), the functions h^k, $k\in P_f^b(x)$, have the nice property that

(3.22) $\quad D_{h^k}^=(x) = \{d\in R^n: \nabla h^k(x;d) = 0\}$.

Furthermore, since $F \subset \underset{\sim}{F}$ and $C_{\underset{\sim}{P}(x)}(x) = C_{P(x)}(x)$, we get that

$$\underset{\sim}{P}^= \subset P^= \quad \text{and} \quad \underset{\sim}{P}_f^b(x) \subset P_f^b(x) \ .$$

Therefore, by (3.22), we see that

$$\mathcal{P}_f^b(x) = \emptyset .$$

Now by Corollary 3.4, we see that

$$D_f^<(x) \cap C_{\mathcal{P}(x)}(x) = \emptyset .$$

But since $C_{\mathcal{P}(x)}(x) = C_{\mathcal{P}(x)}(x)$, we now conclude, by the definition of $\mathcal{P}_f^b(x)$, $\mathcal{P}_f^b(x) = \emptyset$ which yields the desired contradiction. □

4. *Identifying 'Vital' Resources.*

Suppose that x solves (P). We will now show that the set of 'vital' resources of (P) corresponds to the set $\mathcal{P}_f^b(x)$, i.e., an increment in the k^{th} resource, $k \in \mathcal{P}_f^b(x)$ leads to an 'infinite' marginal rate of decrease in the optimal value of the objective function. (Recall that $v(e)$ is the optimal value of the objective function with respect to the perturbation e of the right-hand side of the constraints.)

THEOREM 4.1. *Suppose that* x *solves* (P). *Let* $e = (e_i) \in R^m$ *satisfy*

$$(4.1) \qquad e_i = \begin{cases} 1 & \text{if } i \in \mathcal{P}_f^b(x) , \\ \alpha_i \geq 0 & \text{otherwise.} \end{cases}$$

Then, when $\mathcal{P}_f^b(x) \neq \emptyset$,

$$(4.2) \qquad \nabla v(0;e) = -\infty .$$

Proof. Suppose that $\mathcal{P}_f^b(x) \neq \emptyset$. By Proposition 3.6, there exists

$$(4.3) \qquad d \in \bigcup_{k \in \mathcal{P}_f^b(x)} D_f^<(x) \cap D_k^>(x) \cap C_{\mathcal{P}(x)}(x) \cap D^{\leq}_{\mathcal{P}^<(x)}(x) .$$

Let

$$U_d = \left\{ k \in \mathcal{P}_f^b(x) : d \in D_k^>(x) \right\} .$$

Then, by (4.3),

$$(4.4) \qquad \nabla g^k(x;d) = 0, \quad \text{for all } k \in U_d ,$$

while

$$(4.5) \qquad \nabla f(x;d) = \beta < 0 .$$

Suppose $\alpha_n \downarrow 0$. Then (4.4) implies that

$$0 < \frac{g^k(x+\alpha_n d)}{\alpha_n} \to 0, \quad \text{for all } k \in U_d ,$$

i.e.,

$$0 < g^k(x+\alpha_n d) \leq \alpha_n \beta_n = \epsilon_n, \quad \text{for all } k \in U_d ,$$

where $\beta_n \downarrow 0$. By convexity and $d \in D_k^>(x)$, we get that

$$0 = g^k(x) \leq g^k(x+\alpha d) \leq g^k(x+\alpha_n d) \leq \epsilon_n ,$$

for all $k \in U_d$ and $0 \leq \alpha \leq \alpha_n$. By (4.3) and the definition of $\mathcal{P}_f^b(x)$, we now conclude that, for sufficiently large n,

$$g^k(x+\alpha_n d) \leq \epsilon_n e_k, \quad \text{for all } k \in \mathcal{P} .$$

Therefore

$$\frac{v(\epsilon_n e) - v(0)}{\epsilon_n} \leq \frac{f(x+\alpha_n d) - f(x)}{\epsilon_n}, \quad \text{for sufficiently large } n ,$$

$$= \frac{f(x+\alpha_n d) - f(x)}{\alpha_n \beta_n}$$

$$\to -\infty ,$$

by (4.5) and since $\beta_n \downarrow 0$. □

Note that we do not need to increment all the resources indexed by $\mathcal{P}_f^b(x)$. Indeed, once a d is found which satisfies (4.3), then we need only increment the resources indexed by U_d. In fact, usually U_d will consist of only one element (after discarding redundant constraints).

The next result provides lower bounds for the marginal rate of change of the optimal value of (P) with respect to an increment in the k^{th} resource, $k \notin \mathcal{P}^=$.

THEOREM 4.2. *Suppose that* x *solves* (P). *Using Corollary* 3.1, *choose* $\lambda^* \in R^m$ *an optimal multiplier vector for* (P), *i.e.* $\lambda^* = (\lambda_k^*)$, $\lambda_k^* \geq 0$, $\phi^k \in \partial g^k(x)$, *and* $\sum_{k \in P(x)} \lambda_k^* \phi^k \in B_{P(x)}(x)$ *solves* (3.5). *Then*

$$\nabla v(0;e) \geq - \lambda^* e \ , \tag{4.6}$$

where the perturbation vector $e = (e_i) \in R^m$ *with* $e_i \leq 0$ *for all* $i \in P^=$.

Proof. Corollary 3.1 implies that there exists an optimal multiplier vector $\lambda^* \in R^m$ such that $\lambda^* g(x) = 0$ and

$$f(y) \geq f(x) - \lambda^* g(y) \tag{4.7}$$

for all $y \in x + D^=_{P^=}(x) = x + D^{\leq}_{P^=}(x)$, by (3.10). Therefore, (4.7) holds for all y such that $g^k(y) \leq 0$ for all $k \in P^=$.

Let

$$Z = \{z \in R^m : g(y) \leq z, \text{ for some } y \in R^n\} \ .$$

Then, for each $z = (z_i) \in Z$ with $z_i \leq 0$ for all $i \in P^=$, (4.7) implies

$$f(y) \geq f(x) + (-\lambda^*)z, \text{ for all } y \text{ such that } g(y) \leq z.$$

Taking the infimum on the left-hand side and noting that $f(x) = v(0)$ and $v(z) = \infty$ if $z \notin Z$, yields

$$v(x) - v(0) \geq -\lambda^* z \ ,$$

for all $z = (z_i)$ with $z_i \leq 0$ for all $i \in P^=$. This implies (4.6). □

Remark 4.1. One can replace $D^=_\Omega(x)$ with $D^{\leq}_\Omega(x)$ in Proposition 3.4. Thus, if

$$\left(C_{P(x)}(x) \cap D^{\leq}_\Omega(x)\right)^+ = -B_{P(x)}(x) + \left(D^{\leq}_\Omega(x)\right)^+ \ , \tag{4.8}$$

we can replace $D^=_\Omega(x)$ with $D^{\leq}_\Omega(x)$ in (3.11) of Corollary 3.5 and use this modified form of Corollary 3.5 (instead of Corollary 3.1) in the above theorem to obtain lower bounds for the marginal rate of change of the optimal value of (P) with respect to an increment in the k^{th} resource,

$k \notin \Omega$, i.e., if λ^* is the optimal multiplier vector for (P) obtained using the modified Corollary 3.5, then

$$\nabla v(0;e) \geq -\lambda^* e \ , \tag{4.9}$$

where $e_i \leq 0$ for all $i \in \Omega$. Note that this result is not true in general, if (4.8) fails.

Example 4.1. Suppose that

$$f(x) = x_2; \ g^1(x) = x_1^2 + x_2^2 - 2; \ g^2(x) = -x_1 + \sqrt{2}$$

with $x = (x_i)$ in R^2. Then $x = (\sqrt{2},0)$ is the only feasible point of (P) and thus is optimal with optimal value $f(x) = v(0) = 0$. Moreover, $P(x) = P^= = \{1,2\}$, while $P_f^b(x) = \{1\}$ since $d = (0,-1) \in D_f^<(x) \cap D_1^>(x) \cap C_{P(x)}(x)$. Note that $2 \notin P_f^b(x)$ since g^2 is affine. Now let us consider the perturbation direction

$$e = (0,1) \ .$$

Then, for $t > 0$,

$$v(te) = \inf\{f(y) : g^1(y) \leq 0, \ g^2(y) \leq t\}$$
$$= -\sqrt{2 - (\sqrt{2} - t)^2} = -\sqrt{2\sqrt{2}t - t^2}$$

and

$$\nabla v(0;e) = \lim_{t \downarrow 0} \frac{v(te) - v(0)}{t}$$
$$= \lim_{t \downarrow 0} \frac{-\sqrt{2\sqrt{2}t - t^2} - 0}{t} = - \infty \ .$$

Thus e is an unstable perturbation even though $e_i \leq 0$ for all $i \in \Omega$, with $\Omega = P_f^b(x)$. In addition, Theorem 4.1 implies $\nabla v(0;(1,0)) = - \infty$. Thus both marginal values are infinite. Note that (4.8) fails to hold.

Example 4.2. Suppose that

$$f(x) = -x_1 + x_2 + x_3; \ g^1(x) = x_1^2 + x_2^2 - 2 \ ;$$
$$g^2(x) = -x_1 + 1; \ g^3(x) = -x_2 + 1; \ g^4(x) = -x_3 \ ,$$

with $x = (x_i)$ in R^3. Then the feasible set

$$F = \{x: x_1 = 1,\ x_2 = 1,\ x_3 \geq 0\}$$

while $x = (1,1,0)$ is the optimal solution with optimal value $f(x) = v(0) = 0$. Moreover, $P(x) = P^{=} = \{1,2,3,4\}$ and $P_f^b(x) = \{1\}$. We set

$$\Omega = P_f^b(x) .$$

Then $D_\Omega^{=}(x) = \{d: d_1 = d_2 = 0\}$; $D_\Omega^{\leq}(x) = D_\Omega^{=}(x) \cup D_\Omega^{<}(x) = \{d: d_1 + d_2 \leq 0\}$; $C_{P(x)}(x) = \{d: d_1 = d_2 = 0\},\ d_3 \geq 0\}$;

$$C_{P(x)}(x) \cap D_\Omega^{\leq}(x)^+ = \{d: d_3 \geq 0\} = C_{P(x)}^+(x) + \left(D_\Omega^{\leq}(x)\right)^+ .$$

Thus (4.7) holds. The modified Corollary 3.5 yields the system

(4.9) $$\begin{pmatrix}-1\\1\\1\end{pmatrix} + \lambda_1^*\begin{pmatrix}2\\2\\2\end{pmatrix} + \lambda_2^*\begin{pmatrix}-1\\0\\0\end{pmatrix} + \lambda_3^*\begin{pmatrix}0\\-1\\0\end{pmatrix} + \lambda_4^*\begin{pmatrix}0\\0\\-1\end{pmatrix} \in \{d: d_1 = d_2 \leq 0,\ d_3 = 0\} .$$

A solution must satisfy

$$\lambda_4^* = 1;\quad \lambda_2^* = \lambda_3^* - 2;\quad 1 + 2\lambda_1^* - \lambda_3^* \leq 0 .$$

One particular solution is

$$\lambda^* = (0,0,2,1) .$$

From Theorem 4.1, we get that

$$\nabla v(0;(1,0,0,0)) = -\infty ,$$

i.e., the marginal value for the first constraint is infinite. However, Remark 4.1 gives

(4.10) $$\begin{cases} \nabla v(0,(0,1,0,0)) \geq 0 ; \\ \nabla v(0,(0,0,1,0)) \geq -2; \\ \nabla v(0,(0,0,0,1)) \geq -1. \end{cases}$$

Thus we have marginal values for all constraints. Note that these are the best values obtainable, since they are the smallest nonnegative values which solve (4.9). In fact, equality holds in (4.10).

In summary, if a constraint qualification holds for (P) and $\lambda^* = (\lambda_k^*)$ is a Kuhn-Tucker multiplier vector (obtained using (3.2)), then it is well-known that

(5.1) $$\nabla v(0;e) \geq -\lambda^* e$$

for all perturbation directions $e = (e_k)$. If e is the k^{th} unit vector, this shows that λ_k^* is a marginal value for the constraint g^k. In the absence of a constraint qualification, we can obtain an optimal multiplier vector λ^* using Corollary 3.1. Then (5.1) still holds if the perturbation direction e satisfies $e_k \leq 0$ for all $k \in P^{=}$. Thus λ_k^* is a marginal value for each constraint g^k, $k \notin P^{=}$. If (4.7) holds for some set Ω satisfying (3.9) and an optimal multiplier vector λ^* is obtained using Corollary (3.5) (with $D_\Omega^{\leq}(x)$ replacing $D_\Omega^{=}(x)$ in (3.11)), then again (5.1) holds but now we only need $e_k \leq 0$ for all $k \in \Omega$. Thus the λ_k^* are marginal values for each constraint g^k, $k \notin \Omega$. Finally, if $k \in P_f^b(x)$, then the k^{th} marginal value (assuming that the constraint g^k is not redundant at the optimum point) is infinite.

REFERENCES

[1] R. A. Abrams and L. Kerzner, *A simplified test for optimality*, Journal of Optimization Theory and Applications *25* (1978), 161-170.

[2] A. Ben-Israel, A. Ben-Tal, and S. Zlobec, *Optimality conditions in convex programming*, The IX International Symposium on Mathematical Programming, Budapest, Hungary, August, 1976.

[3] A. Ben-Tal and A. Ben-Israel, *Characterizations of optimality in convex programming: the nondifferentiable case*, Applicable Analysis, *9* (1979), 137-156.

[4] A. Ben-Tal, A. Ben-Israel, and S. Zlobec, *Characterization of optimality in convex programming without a constraint qualification*, Journal of Optimization Theory and Applications *20* (1976), 417-437.

[5] J. M. Borwein and H. Wolkowicz, *Characterizations of optimality without constraint qualification for the abstract convex program*, Journal of the Australian Mathematical Society, to appear.

[6] A. Ya Dubovitskii and A. A. Milyutin, *Extremum problems in the presence of restrictions*, Zh. vychish. Mat. mat. Fiz. *5* (1965), 395-453.

[7] A. M. Geoffrion, *Duality in nonlinear programming: A simplified applications-orientated development*, SIAM Review *13* (1971), 1-37.

[8] O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969.

[9] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[10] R. T. Rockafellar, "Some convex programs whose duals are linearly constrained" in *Nonlinear Programming*, J. B. Rosen, O. L. Mangasarian, K. Ritter, eds., Academic Press, New York, 1970, pp. 293-322.

[11] H. Wolkowicz, *Calculating the cone of directions of contancy*, Journal of Optimization Theory and Applications *25* (1978), 451-457.

[12] H. Wolkowicz, *Constructive approaches to approximate solutions of operator equations and convex programming*, Ph.D. Thesis, McGill University, 1978.

[13] H. Wolkowicz, *Geometry of optimality conditions and constraint qualifications: The convex case*, Mathematical Programming, to appear.

[14] G. Zoutendijk, *Methods of Feasible Directions, A Study in Linear and Non-linear Programming*, Elsevier, New York, 1960.

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