Bounds for Ratios of Eigenvalues Using Traces*

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ABSTRACT

Let A be an $n\times n$ matrix with real eigenvalues $\lambda_1\geqslant\cdots\geqslant\lambda_n$, and let $1\leqslant k<l\leqslant n$. Bounds involving $\operatorname{tr}A$ and $\operatorname{tr}A^2$ are introduced for λ_k/λ_l, $(\lambda_k-\lambda_l)/(\lambda_k+\lambda_l)$, and $(k\lambda_k+(n-l+1)\lambda_l)^2/(k\lambda_k^2+(n-l+1)\lambda_l^2)$. Also included are conditions for $\lambda_l>0$ and for $\lambda_k+\lambda_l>0$.

1. INTRODUCTION

Bounds for the modulus, the real part, and the imaginary part of a linear combination of the ordered eigenvalues of an $n\times n$ complex matrix A were

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obtained in [4], [5]. These bounds used the traces of the matrix A and its square A^2. Throughout this paper we assume that A is nonzero and has real eigenvalues, and $n \geqslant 2$. We will find upper bounds for the ratios

$$\gamma_{kl} = \frac{\lambda_k}{\lambda_l},$$

$$\delta_{kl} = \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l},$$

and

$$\eta_{kl} = \frac{\{k\lambda_k + (n-l+1)\lambda_l\}^2}{k\lambda_k^2 + (n-l+1)\lambda_l^2}.$$

Here $1 \leqslant k < l \leqslant n$, and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are the ordered eigenvalues of A.

In Section 2 we present several preliminary definitions and results including conditions (necessary and/or sufficient) which guarantee that $\lambda_l > 0$ and/or that $\lambda_k + \lambda_l > 0$. These conditions are needed when deriving the bounds for γ_{kl} and δ_{kl}. (A side result (Proposition 2.2) extends the bounds for the average of a set of consecutive eigenvalues obtained in [4, Eq. (2.19)] to the average of a set of noncontiguous eigenvalues.) The bounds for γ_{kl} are presented in Section 3, while those for δ_{kl} and η_{kl} are given in Sections 4 and 5, respectively.

If A is positive definite, then γ_{1n} is the "condition number" of A (e.g., [2]) while δ_{1n} equals the Kantorovich ratio (e.g., [1]). These ratios, as well as γ_{kl} and δ_{kl}, are useful in error and convergence-rate analysis for solutions of systems of equations and mathematical programs (see [1] and [2]).

2. PRELIMINARIES

As in [4, 5], our bounds use the traces

$$\operatorname{tr} A = \sum_{i=1}^{n} \lambda_i = a \tag{2.1a}$$

and

$$\operatorname{tr} A^2 = \sum_{i=1}^{n} \lambda_i^2 = b, \tag{2.1b}$$

where a and b are real numbers. Our bounds, therefore, will hold for any

$n \times n$ complex matrix A which has real eigenvalues and which satisfies (2.1). We now let

$$m = \frac{\operatorname{tr} A}{n} \tag{2.2}$$

and

$$s^2 = \frac{\operatorname{tr} A^2}{n} - m^2. \tag{2.3}$$

Given a and b, the equations (2.1) admit a real solution if and only if $s^2 \geqslant 0$; cf. [6]. We will therefore suppose throughout this paper that $s^2 \geqslant 0$, and we will take s as the nonnegative square root of s^2.

To derive bounds for γ_{kl} and δ_{kl}, we must have $\lambda_l > 0$ and $\lambda_k + \lambda_l > 0$. The following results provide sufficient (and necessary) conditions for this to hold.

PROPOSITION 2.1. *Suppose that* $\operatorname{tr} A \geqslant 0$ *and* $2 \leqslant l \leqslant n$. *Then the following are equivalent:*

(i) *Every* $n \times n$ *matrix* A *with real eigenvalues satisfying* (2.1) *has* $\lambda_l > 0$.
(ii) $(\operatorname{tr} A)^2 > (l-1)\operatorname{tr} A^2$.

Proof. From [4, Equation (2.22)] we have

$$\lambda_l \geqslant m - s\left(\frac{l-1}{n-l+1}\right)^{1/2}, \tag{2.4}$$

with equality if and only if $\lambda_1 = \cdots = \lambda_{l-1}$ and $\lambda_l = \cdots = \lambda_n$. The right-hand side of (2.4) is positive if and only if (ii) holds. ∎

Note that we must assume $a = \operatorname{tr} A \geqslant 0$ in order to guarantee that the right-hand side of (2.4) is positive. If $\operatorname{tr} A < 0$ and $l \geqslant 2$, then there always exists a matrix B such that $\operatorname{tr} B = \operatorname{tr} A$, $\operatorname{tr} B^2 = \operatorname{tr} A^2$, and the lth ordered eigenvalue of B is < 0. However, if $l = 1$, then by [4, Theorem 2.1]

$$\lambda_1 \geqslant m + \frac{s}{(n-1)^{1/2}},$$

with equality if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$. Therefore $\lambda_1 > 0$ if $\operatorname{tr} A \geqslant 0$

and $\operatorname{tr} A^2 \neq 0$. When $\operatorname{tr} A < 0$, then $\lambda_1 > 0$ if

$$(\operatorname{tr} A)^2 = a^2 < b = \operatorname{tr} A^2. \tag{2.5}$$

Condition (ii) above naturally guarantees that $\lambda_k + \lambda_l > 0$ for $k < l$. However, (ii) is a very restrictive condition, particularly when the difference $l - k$ is large. The following corollary improves on condition (ii).

COROLLARY 2.1. *Let $a \geqslant 0$, $1 \leqslant k < l \leqslant n$, and*

$$t = \max(k, 2l - n - 1).$$

A sufficient condition for every ordered n-tuple satisfying (2.1) *also to satisfy*

$$\lambda_k + \lambda_l > 0 \tag{2.6}$$

is that

$$a^2 > (t-1)b. \tag{2.7}$$

The condition (2.7) *is also necessary if $k \geqslant 2$ and*

$$l - k - 1 \leqslant n - l. \tag{2.8}$$

Moreover, if $k = 1$ and $l \leqslant (n+2)/2$, then (2.6) *always holds.*

Proof. Let

$$\lambda_{(g,h)} = \frac{1}{h-g+1} \sum_{j=g}^{h} \lambda_j. \tag{2.9}$$

Since $l > t \geqslant k$,

$$\begin{aligned} \tfrac{1}{2}(\lambda_k + \lambda_l) &\geqslant \tfrac{1}{2}(\lambda_t + \lambda_l) \\ &\geqslant \tfrac{1}{2}\left(\lambda_{(t,l-1)} + \lambda_{(l,2l-t-1)}\right) \\ &= \lambda_{(t,2l-t-1)} \\ &\geqslant \lambda_{(t,n)} \\ &\geqslant m - s\left(\frac{t-1}{n-t+1}\right)^{1/2}, \end{aligned}$$

by [4, Equation (2.19)], with equality in the last inequality if and only if

$\lambda_1 = \cdots = \lambda_{t-1}$ and $\lambda_t = \cdots = \lambda_n$. (This therefore characterizes equality in all the inequalities together except for the first inequality.) The first result now follows, since $m - s((t-1)/(n-t+1))^{1/2} > 0$ if and only if (2.7) holds.

The necessity of (2.7) follows from the above mentioned conditions for equality since equality holds in the first inequality in (2.9) if $t = k$, which is equivalent to (2.8). Finally, if $k = 1$ and $2l - 2 \leqslant n$, then

$$\tfrac{1}{2}(\lambda_1 + \lambda_l) \geqslant \lambda_{(1,2l-2)} \geqslant a/n > 0, \tag{2.10}$$

and the proof is complete. ∎

The inequality (2.7) always improves on condition (ii) in Proposition 2.1 Moreover, Corollary 2.1 also shows that (2.7) is the best possible condition if $l - k - 1 \leqslant n - l$, and the only information we use is the triple (n, a, b). The following corollary provides an alternative sufficient condition which may improve on (2.7) when $l - k - 1 > n - l$.

COROLLARY 2.2. *Suppose $a > 0$ and $1 \leqslant k < l \leqslant n$. If*

$$a^2 > \left(\frac{(n-2)u}{(n-1)^2 - u}\right) b, \tag{2.11}$$

where

$$u = n(k-1) + (l-1)(\tfrac{1}{2}l - k),$$

then every ordered real n-tuple (λ_i) satisfying (2.1) also satisfies

$$\lambda_k + \lambda_l > 0. \tag{2.12}$$

Proof. Consider the diagonal $r \times r$ matrix B with ordered diagonal elements

$$\mu_h = \lambda_i + \lambda_j, \qquad h = 1, \ldots, r, \tag{2.13}$$

where $r = n(n-1)/2$ and $1 \leqslant i < j \leqslant n$. Let

$$a_B = \operatorname{tr} B, \qquad b_B = \operatorname{tr} B^2,$$

$$m_B = \frac{\operatorname{tr} B}{n}, \qquad s_B^2 = \frac{\operatorname{tr} B^2}{n} - m_B^2.$$

There are at least

$$r-q+1=\binom{n-l+2}{2}+(l-k-1)(n-l+1) \tag{2.14}$$

$(\lambda_i+\lambda_j)$'s less than or equal to $\lambda_k+\lambda_l$, namely those for which i and j satisfy

$$\begin{gathered} i=k, \qquad j=l,\dots,n; \\ l\leqslant i<j\leqslant n; \\ k<i<l\leqslant j\leqslant n. \end{gathered} \tag{2.15}$$

Therefore [cf. (2.4) and (2.14)],

$$\begin{aligned} \lambda_k+\lambda_l \geqslant \mu_q \geqslant m_B - s_B\left(\frac{q-1}{r-q+1}\right)^{1/2} \\ = m_B - s_B\left(\frac{u}{r-u}\right)^{1/2}. \end{aligned} \tag{2.16}$$

Now since

$$a_B=(n-1)a \quad \text{and} \quad b_B=a^2+(n-2)b,$$

we obtain, using (2.11),

$$m_B - s_B\left(\frac{u}{r-u}\right)^{1/2} > 0,$$

which implies $\lambda_k+\lambda_l>0$, using (2.16), if and only if

$$m_B^2 > s_B^2\left(\frac{u}{r-u}\right)$$

if and only if

$$a^2 > \frac{ra^2+r(n-2)b-(n-1)^2a^2}{(n-1)^2}\,\frac{u}{r-u} \tag{2.17}$$

if and only if (2.11). ■

The simple case $n = 2$ can be included in the above, but can also be studied separately. Here, $k = 1$ and $l = 2$. Therefore, since $\lambda_1 + \lambda_2 = \operatorname{tr} A$, (2.11) and (2.12) are always valid if $\operatorname{tr} A > 0$.

Let $2 \leqslant l \leqslant p \leqslant n$ and $l \leqslant q \leqslant (n+l+1)/2$. Now

$$\lambda_l \geqslant \lambda_{(l,p)} \geqslant \lambda_{(l,n)}, \tag{2.18}$$

$$\tfrac{1}{2}(\lambda_l + \lambda_q) \geqslant \lambda_{(l,n)}, \tag{2.19}$$

and by [4, Theorem 2.2]

$$\lambda_{(l,p)} \geqslant m - s\left(\frac{l-1}{n-l+1}\right)^{1/2}, \tag{2.20}$$

with equality throughout (2.18), (2.19), and (2.20) if and only if

$$\lambda_1 = \cdots = \lambda_{l-1} \quad \text{and} \quad \lambda_l = \cdots = \lambda_n.$$

Since equality can be attained, we see that

$$\lambda_l > 0 \quad \Leftrightarrow \quad \sum_{i=l}^{p} \lambda_i > 0 \quad \Leftrightarrow \quad \sum_{i=l}^{n} \lambda_i > 0 \quad \Leftrightarrow \quad \lambda_l + \lambda_q > 0. \tag{2.21}$$

Thus we have the following four interesting conditions equivalent to (i) and (ii) in Proposition 2.1:

(iii) For a *fixed* p, $2 \leqslant l \leqslant p \leqslant n$, every ordered real n-tuple (λ_i) satisfying (2.1) also satisfies $\lambda_l + \cdots + \lambda_p > 0$.

(iv) For *any* p, $2 \leqslant l \leqslant p \leqslant n$, every ordered real n-tuple (λ_i) satisfying (2.1) also satisfies $\lambda_l + \cdots + \lambda_p > 0$.

(v) For a *fixed* q, $2 \leqslant l \leqslant q \leqslant (n+l+1)/2$, every ordered real n-tuple (λ_i) satisfying (2.1) also satisfies $\lambda_l + \lambda_q > 0$.

(vi) For *any* q, $2 \leqslant l \leqslant q \leqslant (n+l+1)/2$, every ordered real n-tuple (λ_i) satisfying (2.1) also satisfies $\lambda_l + \lambda_q > 0$.

Nonconsecutive eigenvalues λ_l, λ_q for which $q > (n+l+1)/2$ behave differently. For example let $n = 5$, $a = 8$, and $b = 52$. By (ii) in Proposition 2.1, every n-tuple (λ_i) satisfying (2.1) has $\lambda_2 > 0$, and by (iv) and (vi) above, also $\lambda_2 + \lambda_3 > 0$, $\lambda_2 + \lambda_3 + \lambda_4 > 0$, $\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 > 0$, $\lambda_2 + \lambda_4 > 0$.

However, $(3,3,3,3,-4)$ satisfies (2.1) with $a = 8$ and $b = 52$ as above, but has $\lambda_2 + \lambda_5 < 0$. Note that $q = 5 > (5+2+1)/2 = (n+l+1)/2$.

Proposition 2.1 and its two corollaries give sufficient conditions for $\lambda_k + \lambda_l > 0$ $(k < l)$. To compare these, we have to compare

$$l-1, \qquad t-1, \quad \text{and} \quad \frac{(n-2)u}{(n-1)^2-u} = f,$$

where $t = \max(k, 2l-n-1)$ and $u = n(k-1)+(l-1)(\frac{1}{2}l-k)$. Since $k < l \leqslant n$, it follows at once that $l-1 > t-1$.

We now compare $t-1$ with f. Let

$$y = t - 1 - f.$$

Then (2.7) is less restrictive than (2.11) whenever $y < 0$. From (2.8) in Corollary 2.1, when

$$x = (l-k-1)-(n-l) = 2l-k-n-1$$

is less than or equal to zero, then $y \leqslant 0$. This is illustrated in Figure 1, where we have plotted the region in the (x, y)-plane for the $\frac{1}{2}n(n-1) = 1225$ values of x and y with $1 \leqslant k < l \leqslant n = 50$. From Figure 1 we also observe that $y \leqslant 0$ whenever $x \leqslant 11$ and that $y \geqslant 0$ whenever $x \geqslant 24$. Additional computations for $n = 2(1)100$ show that $y \geqslant 0$ whenever $x \geqslant [\frac{1}{2}n]-1$, where $[\cdot]$ denotes the integer part. We have not yet, however, been able to prove this observation.

One can generalize the above procedures in order to find bounds (upper and lower) for

$$\lambda_{[i_1,\ldots,i_k]} = \frac{1}{k}\sum_{j=1}^{k}\lambda_{i_j},$$

which is the average of k noncontiguous eigenvalues $(1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n)$. When the eigenvalues are contiguous we write [cf. (2.9)]

$$\lambda_{(g,h)} = \frac{1}{h-g+1}\sum_{j=g}^{h}\lambda_j.$$

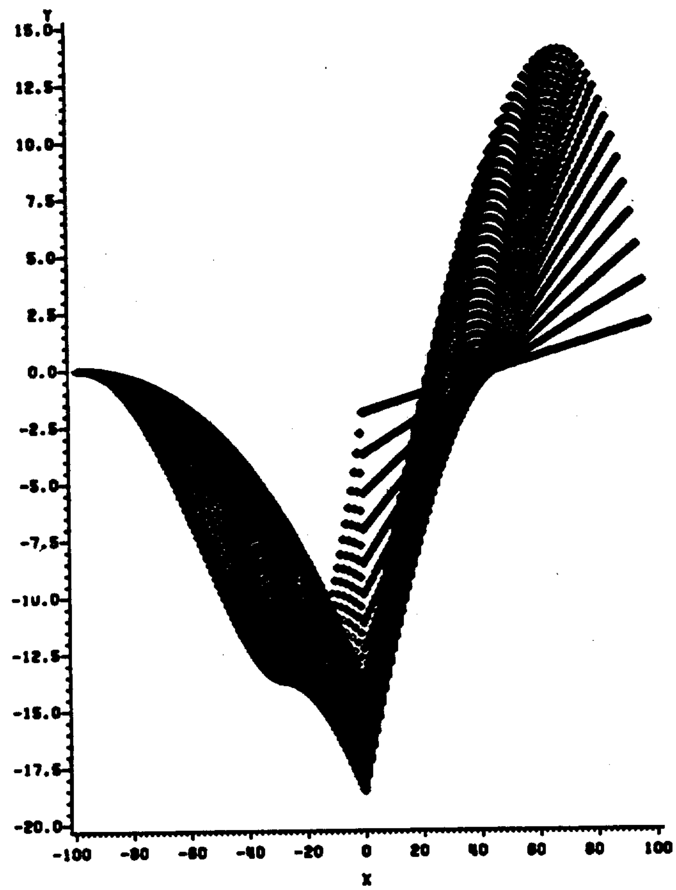

FIG. 1. The "bird" when $n = 50$.

PROPOSITION 2.2. *Suppose that* $\operatorname{tr} A \geq 0$. *Set*

$$r_0 = 0; \qquad r_1 = i_1; \qquad r_j = \min(i_j, 2r_{j-1} - r_{j-2}), \quad j = 2, \dots, k;$$

$$t_{k+1} = n + 1; \qquad t_k = i_k; \qquad t_j = \max(i_j, 2t_{j+1} - t_{j+2}), \quad j = k-1, \dots, 1;$$

$$r \equiv r_k; \qquad t \equiv t_1.$$

Then

$$m - s\left(\frac{t-1}{n-t+1}\right)^{1/2} \leq \lambda_{[i_1, \dots, i_k]} \leq m + s\left(\frac{n-r}{r}\right)^{1/2}. \tag{2.24}$$

Proof. Since $r_j \leqslant i_j$ $(j = 1,\dots,k)$, we have

$$\lambda_{[i_1,\dots,i_k]} \leqslant \lambda_{[r_1,\dots,r_k]} \leqslant \lambda_{(1,r)},$$

as $r_j - r_{j-1} \leqslant r_{j-1} - r_{j-2}$ $(j = 2,\dots,k)$. Then the right-hand inequality in (2.24) follows at once from [4, Equation (2.24) *sic*]. The left-hand inequality follows similarly. ∎

Conditions characterizing equality on the left (and on the right) of (2.24) may be obtained directly from [4, Theorem 2.2]. Improvements to (2.24) occur when $i_1 = 1$ $(i_k = n)$ and may be obtained by setting $l = i_k$ in [4, Equation (2.25)] $(k = i_1$ in [4, Equation (2.26)]).

The second approach involves considering the $\binom{n}{p} \times \binom{n}{p}$ diagonal matrix —call it D—with diagonal elements consisting of all the possible sums of p elements $\sum_{j=1}^{p} \lambda_{k_j}$. Then we need to express $\operatorname{tr} D$ and $\operatorname{tr} D^2$ in terms of $\operatorname{tr} A$ and $\operatorname{tr} A^2$ and use the known bounds for the ordered eigenvalues of D.

3. UPPER BOUND FOR γ_{kl}

THEOREM 3.1. *Let* $1 \leqslant k < l \leqslant n$, $\operatorname{tr} A \geqslant 0$, *and*

$$(l-1)\operatorname{tr} A^2 < (\operatorname{tr} A)^2. \tag{3.1}$$

Then

$$\lambda_l > 0 \tag{3.2}$$

and

$$\gamma_{kl} \leqslant \frac{c + k + \left\{ \dfrac{n-l+1}{k}(c+k)(n-l+1-c) \right\}^{1/2}}{c + k - \left\{ \dfrac{k}{n-l+1}(c+k)(n-l+1-c) \right\}^{1/2}}, \tag{3.3}$$

where

$$c = \frac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (l-1). \tag{3.4}$$

Equality holds if and only if

$$\lambda_1 = \cdots = \lambda_k,$$
$$\lambda_{k+1} = \cdots = \lambda_{l-1} = \frac{\operatorname{tr} A^2}{\operatorname{tr} A}, \tag{3.5}$$
$$\lambda_l = \cdots = \lambda_n.$$

Proof. That (3.2) holds follows from Proposition 2.1. We prove the rest by solving

Problem A. Maximize $\gamma_{kl} = \lambda_k / \lambda_l$ subject to

$$\lambda_1 \geqslant \cdots \geqslant \lambda_n, \tag{3.6}$$
$$\lambda_1 + \cdots + \lambda_n = a, \tag{3.7}$$
$$\lambda_1^2 + \cdots + \lambda_n^2 \leqslant b, \tag{3.8}$$
$$\lambda_l > 0, \tag{3.9}$$

where the λ's are the variables, while $a = \operatorname{tr} A$ and $b = \operatorname{tr} A^2$.

Thus we know that $a^2 \leqslant nb$, i.e., there exists at least one solution to (3.6)–(3.9) (see Proposition 2.1). Moreover, we can eliminate the trivial case $a^2 = nb$, since this holds if and only if $\lambda_1 = \cdots = \lambda_n$.

First, let us show that equality must hold in (3.8). Suppose not, and let h_k denote the multiplicity of λ_k and h_l denote the multiplicity of λ_l. (Allocate the multiplicities arbitrarily, though consistently with the ordering, if $\lambda_k = \lambda_l$.) Let $d > 0$, and perturb the h_k λ_i's equal to λ_k to $\lambda_k + d/h_k$, and the h_l λ_i's equal to λ_l to $\lambda_l - d/h_l$. The perturbed λ_i's satisfy (3.6)–(3.9) for sufficiently small $d > 0$. Since the value of γ_{kl} is increased, we have obtained a contradiction. Thus we must have

$$\lambda_1^2 + \cdots + \lambda_n^2 = b. \tag{3.10}$$

We employ the perturbation technique repeatedly in order to obtain the solution of Problem A. Now let us show that

$$\lambda_1 = \cdots = \lambda_k, \tag{3.11}$$
$$\lambda_{k+1} = \cdots = \lambda_{l-1}, \tag{3.12}$$
$$\lambda_l = \cdots = \lambda_n. \tag{3.13}$$

Suppose that (3.11)–(3.13) do not hold, but that (λ_i) solves Problem A, that λ_k has multiplicity h_k, and that there exist at least two other distinct eigenvalues $\mu > \nu$, neither equal to λ_l. We perturb all of the h_k λ_k's to $\lambda_k + d$, where $d > 0$, the μ to $\mu + x$, and the ν to $\nu - h_k d - x$. The perturbed λ_i's satisfy (3.7). The condition (3.8) is satisfied if

$$h_k(\lambda_k + d)^2 + (\mu + x)^2 + (\nu - h_k d - x)^2 < h_k\lambda_k^2 + \mu^2 + \nu^2. \quad (3.14)$$

If d and $|x|$ are sufficiently small, we can omit the terms of order two in d and $|x|$. Then (3.14) is valid if

$$\frac{x}{d} > \frac{h_k(\nu - \lambda_k)}{\mu - \nu}. \quad (3.15)$$

Since $\mu > \nu$, the inequality (3.15) always has a solution with d and $|x|$ as small as desired. Thus we can solve (3.6)–(3.9) and increase γ_{kl}, a contradiction. Therefore we have at most three distinct values in the solution.

Now suppose that $\lambda_1 > \lambda_k$. Then we perturb the h_k λ_k's to $\lambda_k + d$, where $d > 0$, and λ_1 to $\lambda_1 - h_k d$. Then for sufficiently small $d > 0$,

$$\begin{aligned}(\lambda_1 - h_k d)^2 + h_k(\lambda_k + d)^2 &= \lambda_1^2 + h_k\lambda_k^2 - 2h_k d(\lambda_1 - \lambda_k) + h_k(h_k + 1)d^2 \\ &< \lambda_1^2 + h_k\lambda_k^2,\end{aligned}$$

i.e., (3.8) is satisfied for small $d > 0$. Since (3.6)–(3.9) are now satisfied for small $d > 0$, while γ_{kl} is increased, we have a contradiction. This proves (3.11). Similarly (3.13) holds.

To prove (3.12), suppose $\lambda_k = \lambda_{k+1}$. We perturb $\lambda_1 = \lambda_k$ to $\lambda_1 + d$ $(d > 0)$ and λ_{k+1} to $\lambda_{k+1} - kd - x$. Denoting by j the smallest integer greater than $k+1$ such that $\lambda_j = \lambda_l$, we also perturb λ_j to $\lambda_j + x$. (Note that $\lambda_{k+1} > \lambda_l$, for if not we deduce that $\lambda_1 = \cdots = \lambda_n$.) The perturbed λ_i's satisfy (3.7), and for sufficiently small d and $|x|$ they also satisfy (3.9), and (3.8) also if in addition

$$\frac{x}{d} > \frac{k(\lambda_k - \lambda_{k+1})}{\lambda_{k+1} - \lambda_j} = 0 \qquad (x > 0). \quad (3.16)$$

If $j < l$, the perturbed λ_i's increase γ_{kl}, which yields a contradiction. If $j = l$, let $\alpha = x/d$. Then we now have the new γ_{kl} equal to

$$\frac{\lambda_k + d}{\lambda_l + \alpha d} = f(d) \qquad (\alpha \text{ fixed}). \quad (3.17)$$

Since $f'(d) > 0$ if $\alpha < \lambda_l/\lambda_k$, there exist α, d such that $f(d) > f(0)$, i.e., γ_{kl} is increased again. Thus we cannot have $\lambda_k = \lambda_{k+1}$. Similarly, we can show that $\lambda_l = \lambda_{l-1}$ leads to a contradiction. This completes the proof that (3.11)–(3.13) must hold. (Recall that we have shown there are at most three distinct λ_i's.)

Now let $x = \lambda_k$, $z = \lambda_{k+1}$, and $y = \lambda_l$. Then (3.11)–(3.13) imply that

$$x \geqslant z \geqslant y, \tag{3.18}$$

$$kx + (l-k-1)z + (n-l+1)y = a, \tag{3.19}$$

$$kx^2 + (l-k-1)z^2 + (n-l+1)y^2 = b, \tag{3.20}$$

$$y > 0, \tag{3.21}$$

while

$$\gamma_{kl} = x/y.$$

Let

$$a_1 = a - (l-k-1)z, \qquad b_1 = b - (l-k-1)z^2.$$

Then (3.19) and (3.20) become

$$kx + (n-l+1)y = a_1, \tag{3.22}$$

$$kx^2 + (n-l+1)y^2 = b_1. \tag{3.23}$$

Eliminating y yields

$$kx^2 + (n-l+1)\left(\frac{a_1 - kx}{n-l+1}\right)^2 - b_1 = 0,$$

or

$$(n+k-l+1)kx^2 - 2ka_1x + a_1^2 - (n-l+1)b_1 = 0,$$

which implies

$$x = \frac{2ka_1 \pm D^{1/2}}{2(n+k-l+1)k}, \tag{3.24}$$

where

$$D = 4k(n-l+1)\{(n+k-l+1)b_1 - a_1^2\}.$$

Similarly, eliminating x yields

$$y = \frac{2(n-l+1)a_1 \pm D^{1/2}}{2(n-l+1)(n+k-l+1)}. \tag{3.25}$$

Since we are maximizing γ_{kl}, we choose the positive root in (3.24) and the negative root in (3.25), i.e., we want to maximize

$$\frac{x}{y} = \frac{2ka_1 + D^{1/2}}{2(n-l+1)a_1 - D^{1/2}} \frac{n-l+1}{k} \tag{3.26}$$

over all z for which the discriminant $D > 0$. Let

$$f(z) = \frac{2ka_1 + D^{1/2}}{2(n-l+1)a_1 - D^{1/2}}. \tag{3.27}$$

Differentiating, we find that the numerator of $f'(z)$ is

$$N = \frac{1}{D^{1/2}}(n-l+k+1)\left(a_1\frac{dD}{dz} - 2D\frac{da_1}{dz}\right).$$

Substituting for dD/dz and da_1/dz yields that

$$f'(z) = 0 \quad \text{if and only if} \quad b_1 - za_1 = 0, \tag{3.28}$$

which is equivalent to

$$z = b/a. \tag{3.29}$$

Now since

$$\frac{x}{y} = f(z)\frac{n-l+k}{k} \geqslant 1,$$

with equality when $D = 0$—i.e., when

$$z = Z_1 = m + s\left(\frac{n-l+k+1}{l-k-1}\right)^{1/2}$$

or

$$z = Z_2 = m - s\left(\frac{l-k-2}{n-l+k}\right)^{1/2}$$

—and since $Z_2 \leqslant z = b/a \leqslant Z_1$ is the only stationary point of $f(z)$, we conclude that $f(b/a)$ must be the maximum

$$c = \frac{a^2}{b} - (l-1). \tag{3.30}$$

Substituting in (3.24) and (3.25) yields

$$x = \frac{c+k+\left\{\frac{n-l+1}{k}(c+k)(n-l+1-c)\right\}^{1/2}}{n+k-l+1}\frac{b}{a}, \tag{3.31}$$

$$y = \frac{c+k-\left\{\frac{k}{n-l+1}(c+k)(n-l+1-c)\right\}^{1/2}}{n+k-l+1}\frac{b}{a}, \tag{3.32}$$

and also

$$z = \frac{kx^2+(n-l+1)y^2}{kx+(n-l+1)y} = \frac{b}{a}. \tag{3.33}$$

This solves Problem A and yields (3.3). ∎

REMARK 3.1. If $k > 1$ (or if $l < n$), then the minimum of γ_{kl} is 1 and is attained for $\lambda_1 = \cdots = \lambda_{k-1}$ and $\lambda_k = \cdots = \lambda_n$ (or for $\lambda_1 = \cdots = \lambda_l$ and $\lambda_{l+1} = \cdots = \lambda_n$). However, since $f(z)$ has only the one critical point which is a maximum, the minimum values for $k = 1$ and $l = n$ occur at the end points, i.e., $\lambda_1 = m + s(n-1)^{1/2}$ [or $m + s/(n-1)^{1/2}$] and $\lambda_n = m - s(n-1)^{1/2}$ [or

$m - s/(n-1)^{1/2}]$. Comparing the two yields,

$$\gamma_{1n} \geq \frac{m + s(n-1)^{1/2}}{m - s/(n-1)^{1/2}}, \tag{3.34}$$

with equality if and only if $\lambda_2 = \cdots = \lambda_n$.

REMARK 3.2. If $l = k+1$, we get from (3.3)

$$\gamma_{k,k+1} \leq \frac{m + s\left(\dfrac{n-k}{k}\right)^{1/2}}{m - s\left(\dfrac{n-k}{k}\right)^{-1/2}},$$

with equality if and only if $\lambda_1 = \cdots = \lambda_k$ and $\lambda_{k+1} = \cdots = \lambda_n$.

4. UPPER BOUND FOR δ_{kl}

We now use Theorem 3.1 to find an upper bound for δ_{kl}. The bound is given in

THEOREM 4.1. *Let* $1 \leq k < l \leq n$, $\operatorname{tr} A \geq 0$, *and* $t = \max(k, 2l - n - 1)$. *If*

$$(t-1)\operatorname{tr} A^2 < (\operatorname{tr} A)^2 \tag{4.1}$$

(or if (2.11) *holds), then*

$$\lambda_k + \lambda_l > 0 \tag{4.2}$$

and

$$\delta_{kl} \leq \frac{((c+k)(n-l+1-c))^{1/2}(n-l+1+k)}{2(c+k)(k(n-l+1))^{1/2} + ((c+k)(n-l+1-c))^{1/2}(n-l+1-k)}, \tag{4.3}$$

where

$$c = \frac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (l-1).$$

Equality holds if and only if (3.5) *holds.*

Proof. That (4.2) holds follows from Corollaries 2.1 and 2.2. The inequality (4.2) guarantees that δ_{kl} is well defined. Otherwise, we could only conclude that $\delta_{kl} \leqslant \infty$. We now maximize δ_{kl} subject to the same constraints as in Problem A [with (4.2) replacing (3.9)] given in the proof of Theorem 2.1. However (suppressing the indices k and l and assuming $\lambda_l \neq 0$),

$$\delta = f(\gamma) = \frac{\gamma - 1}{\gamma + 1}, \tag{4.4}$$

and $f'(\gamma) > 0$ ($\gamma \neq -1$). All the arguments in the proof of Theorem 2.1 now hold with δ replacing γ in the appropriate places. By (4.2), δ is well defined and positive. Thus δ attains its maximum if and only if (3.5) holds, and (4.3) follows from substituting the right-hand side of (3.3) into (4.4). ■

REMARK 4.1. If $k > 1$ (or $l < n$), the best lower bound for δ_{kl} is 0. This can be improved if $k = 1$ and $l = n$, as was done in Remark 3.1. We get, when $\operatorname{tr} A > 0$,

$$\delta_{1n} \geqslant \frac{ns}{2m(n-1)^{1/2} + s(n-2)},$$

with equality if and only if $\lambda_2 = \cdots = \lambda_n$.

5. LOWER BOUND FOR η_{kl}

The above technique can be applied to any function $f(\gamma_{kl})$ which is monotonic in γ_{kl}. Consider

$$\eta_{kl} = f(\gamma_{kl}) = \frac{\{k\gamma_{kl} + (n-l+1)\}^2}{k\gamma_{kl}^2 + (n-l+1)}. \tag{5.1}$$

Theorem 5.1. *Let* $1 \leqslant k < l \leqslant n$, $k\lambda_k + (n-l+1)\lambda_l > 0$, *and* $\operatorname{tr} A \geqslant 0$. *Then*

$$\eta_{kl} \geqslant \frac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (l-k-1). \tag{5.2}$$

Equality holds if and only if (3.5) *holds.*

Proof. Now (suppressing the indices k and l)

$$f'(\gamma) = \frac{2k(k\gamma + n - l + 1)(n - l + 1)(1-\gamma)}{(k\gamma^2 + n - l + 1)^2}.$$

Then $f'(\gamma) \leqslant 0$, since $\gamma \geqslant 1$ if $\lambda_l > 0$ and $k\gamma + n - l + 1 < 0$ if $\lambda_l < 0$. Therefore, the arguments in the proof of Theorem 2.1 hold again, and we have that $\eta = f(\gamma)$ is a minimum if and only if (3.5) holds. This yields (5.2). ∎

Remark 5.1. If $k = 1$ and $l = n$, we get

$$\frac{(\lambda_1 + \lambda_n)^2}{\lambda_1^2 + \lambda_n^2} \geqslant \frac{(\operatorname{tr} A)^2}{\operatorname{tr} A^2} - (n-2).$$

This always holds (see [4, Lemma 2.3]).

6. CONCLUSIONS

In this paper, we have derived upper bounds for the ratios γ_{kl}, δ_{kl}, and η_{kl}. These bounds were initially obtained using the Kuhn-Tucker optimality conditions of mathematical programming (see [3]). For example, to find the upper bound for $\gamma_{kl} = \lambda_k / \lambda_l$, one explicitly solves the optimization problem

$$\max\Big\{\lambda_k/\lambda_l \colon \sum \lambda_i = \operatorname{tr} A,\ \sum \lambda_i^2 = \operatorname{tr} A^2,\ \lambda_k \leqslant \lambda_i,\ \lambda_j \leqslant \lambda_l,\quad i = 1,\ldots,k,\ j = l,\ldots,n\Big\}.$$

Once the solution is obtained, more elementary proofs may be obtained such as the perturbation techniques used herein. In fact, this perturbation technique is an alternative to the Kuhn-Tucker conditions—i.e., under certain regularity conditions, it can be shown that the Kuhn-Tucker conditions hold if and only if a feasible perturbation of decrease cannot be found.

The eigenvalue bounds use only $\operatorname{tr} A$, $\operatorname{tr} A^2$, and n, and so depend directly on the eigenvalues and not on the particular matrix, i.e., we get the same bounds for A as for UAU^*, where U is any unitary matrix. Thus, in the following example, we do not write the matrix A down explicitly, but rather just write the eigenvalues, $\operatorname{tr} A$, $\operatorname{tr} A^2$, n, and the bounds obtained.

EXAMPLE 6.1. Let $n = 5$, and consider a 5×5 complex matrix A with real eigenvalues

$$\lambda_1 = 5.3, \quad \lambda_2 = 4.3, \quad \lambda_3 = 3.5, \quad \lambda_4 = 2.6, \quad \lambda_5 = 2.5.$$

Then $a = \operatorname{tr} A = 18.2$ and $b = \operatorname{tr} A^2 = 71.84$. Hence $(\operatorname{tr} A)^2/\operatorname{tr} A^2 = a^2/b = 4.61 > 4$, and so by Proposition 2.1 we find that any complex Hermitian matrix A with $\operatorname{tr} A = 18.2$ and $\operatorname{tr} A^2 = 71.84$ must, therefore, be positive definite. (See also [4, Theorem 2.6].) Let

$$\gamma_{kl} = \frac{\lambda_k}{\lambda_l},$$

$$\delta_{kl} = \frac{\lambda_k - \lambda_l}{\lambda_k + \lambda_l},$$

and

$$\eta_{kl} = \frac{(k\lambda_k + (n-l+1)\lambda_l)^2}{k\lambda_k^2 + (n-l+1)\lambda_l^2}.$$

Then (3.3) and (4.3) provide upper bounds for γ_{kl} and δ_{kl}, respectively, while (5.2) gives a lower bound for η_{kl}. In Table 1 we present values of these bounds along with γ_{kl}, δ_{kl}, and η_{kl}, for $1 \leqslant k < l \leqslant 5$. A measure of performance or "efficiency" of our bounds is the ratio of the actual value and the bound, with the larger number in the denominator. With this measure we notice that our bounds are best whenever $k = 1$ and $l = 4$, worst for γ_{kl} and δ_{kl} when $k = 4$ and $l = 5$, and worst for η_{kl} when $k = 1$ and $l = 5$. Moreover, the bound for η_{kl} consistently outperforms the bounds for γ_{kl} and δ_{kl}; in fact the efficiency of (5.2) as a bound for η_{kl} never falls below 90%.

TABLE 1

k,l	γ_{kl}	(3.3)	$\gamma_{kl}/(3.3)$	δ_{kl}	(4.3)	$\delta_{kl}/(4.3)$	η_{kl}	(5.2)	$(5.2)/\eta_{kl}$
1,2	1.233	1.850	0.666	0.104	0.298	0.349	4.961	4.611	0.929
1,3	1.514	1.936	0.782	0.205	0.319	0.643	3.850	3.611	0.938
1,4	2.038	2.127	0.958[a]	0.342	0.360	0.950[a]	2.650	2.611	0.985[a]
1,5	2.120	2.934	0.723	0.359	0.492	0.730	1.772	1.611	0.909[b]
2,3	1.229	1.777	0.692	0.103	0.279	0.369	4.950	4.611	0.932
2,4	1.654	1.978	0.836	0.246	0.328	0.750	3.771	3.611	0.958
2,5	1.720	2.804	0.613	0.265	0.474	0.559	2.850	2.611	0.916
3,4	1.346	1.921	0.701	0.148	0.315	0.470	4.903	4.611	0.940
3,5	1.400	2.758	0.508	0.167	0.468	0.357	3.930	3.611	0.919
4,5	1.040	2.734	0.380[b]	0.020	0.464	0.043[b]	5.000	4.611	0.922

[a]Column maximum.
[b]Column minimum.

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