Dimensionality of Biinfinite Systems

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ABSTRACT

The problem of determining the uniqueness of the coefficient of interpolation of M compactly supported real functions, with a biinfinite sequence of interpolation points, leads to the study of the kernel Z of the biinfinite block Toeplitz matrix

$$D=\begin{bmatrix} \ddots & \ddots & & & \\ & A & B & & \\ & & A & B & \\ & & & \ddots & \ddots \end{bmatrix}.$$

The dimension of Z is found by considering the "maximal solvable subspace" V (relative to A and B). Further results are obtained using the Kronecker canonical form of the matrix pencil $A+\lambda B$ and "restricted generalized inverses" of A (and B).

1. INTRODUCTION

Let $\phi_1,\ldots,\phi_M$ be compactly supported real valued functions, and let a periodic sequence of interpolation points be given satisfying

$$x_i < x_{i+1} \quad \text{and} \quad x_{i+N} = 1 + x_i. \tag{1.1}$$

*Research partially supported by U.S. Army grant DAAG 29-81-K-0013.
**Research partially supported by the Natural Sciences and Engineering Research Council Canada A3388.

0024-3795/84/$3.00

Set

$$G(x) = \sum_{j=-\infty}^{\infty} \sum_{l=1}^{M} a_{jl}\phi_l(x-j).$$

Then one problem of interest concerns the solvability of the interpolation problem

$$G(x_i) = y_i, \qquad -\infty < i < \infty. \tag{1.2}$$

In addition, it is of interest to know if the coefficients α_{jl} are uniquely determined by bounded data when they are restricted to lie in l^{∞}. That is, one must study the block Toeplitz matrix

$$D = \begin{bmatrix} & & & \ddots & \ddots & \ddots & & & \\ \cdot & \cdot & \cdot & A_1 & A_0 & A_{-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & A_1 & A_0 & A_{-1} & \cdot & \cdot \\ & & & & \ddots & \ddots & \ddots & & \end{bmatrix}, \tag{1.3}$$

where

$$A_i = \begin{bmatrix} \phi_1(x_1+i) & \cdots & \phi_M(x_1+i) \\ \vdots & & \vdots \\ \phi_1(x_N+i) & \cdots & \phi_M(x_N+i) \end{bmatrix}.$$

Since the functions have compact support, there exists a t such that $A_i = 0$ if $|i| \geq t$. Thus the matrix D may be reblocked in the form

$$D = \begin{bmatrix} \ddots & \ddots & & & \\ & A & B & & \\ & & A & B & \\ & & & \ddots & \ddots \end{bmatrix}.$$

Of particular interest is the case when the ϕ_l are normalized B-splines with knots $\{t_j\}$ chosen to form an M-periodic sequence:

$$t_j < t_{j+1} \quad \text{and} \quad t_{j+M} = 1 + t_j.$$

Such block Toeplitz collocation matrices are totally positive and have been studied in [2, 7].

We intend to characterize when such matrices are onto or 1-1 either as maps from $l^\infty \to l^\infty$ or from $S \to S$, where S is the space of all sequences. To do this, we study the kernel Z of the block Teoplitz matrix D and the transpose D^t. More precisely, we show how to find dim Z, the dimension of Z, and characterize when dim Z is finite using the "maximal solvable subspace" (relative to A and B).

Section 2 shows that the maximal solvable subspace V is "deflating" if and only if dim Z is finite. In this case, $\dim Z = \dim V$ and each $z = (z_i) \in Z$ is generated by some $x_0 \in V$. Section 3 uses the "Kronecker canonical form" of the matrix pencil $A + \lambda B$ to find dim Z. We see that dim Z is finite if and only if $A + \lambda B$ has no "column singularities." In this case, dim Z is equal to the number of nonzero, finite, generalized eigenvalues of $A + \lambda B$. Section 4 uses restricted generalized inverses to find the maximal solvable subspace. An algorithm based on Gaussian elimination is presented. Several examples are given in Section 5.

2. THE DIMENSION OF Z

We are interested in the dimension of the subspace

$$Z = \{(\dots, x_{-1}, x_0, x_1, \dots)^t : x_i \in C^n \text{ and } Ax_i + Bx_{i+1} = 0\}, \tag{2.1}$$

where A and B are complex $m \times n$ matrices and $\cdot^t$ denotes transpose. Thus Z is the null space (consisting of biinfinite vectors) of the (not necessarily square) block Toeplitz system

$$D = \begin{bmatrix} \ddots & \ddots & & & \\ & A & B & & \\ & & A & B & \\ & & & A & B \\ & & & & \ddots & \ddots \end{bmatrix} \tag{2.2}$$

We call a subspace V of C^n *solvable* (relative to A and B) if

$$AV = BV\,(\subset C^m). \tag{2.3}$$

Recall that a subspace V of C^n is *invariant* (relative to A) if $AV \subset V$. The notion of a solvable subspace is a generalization of a particular invariant subspace. In fact, if we set $B = I$, then the solvable subspaces correspond to the Jordan blocks of A which are not nilpotent. (For a discussion of the invariant subspaces of A, see e.g. [6].)

We now find Z and its dimension by considering the maximum solvable subspace (relative to A and B).

THEOREM 2.1. *The dimension of Z is finite if and only if the maximum solvable subspace V, relative to A and B, satisfies*

$$\dim AV = \dim V\,(=r). \tag{2.4}$$

In this case

$$\dim Z = r. \tag{2.5}$$

Proof. Let $(\dots, z_{-1}, z_0, z_1, \dots)^t$ be any element of Z. Then, the subspace

$$U = \operatorname{span}\{z_i : -\infty < i < \infty\} \tag{2.6}$$

is a solvable subspace, since

$$\begin{aligned} A\left(\sum \alpha_j z_j\right) &= \sum \alpha_j A z_j \\ &= \sum \alpha_j(-Bz_{j+1}) \\ &= B\left(\sum -\alpha_j z_{j+1}\right). \end{aligned} \tag{2.7}$$

Now suppose that we partially order all the solvable subspaces by set inclusion. If V_1 and V_2 are solvable (i.e., $AV_i = BV_i$, $i = 1,2$), then $A(V_1 + V_2) = B(V_1 + V_2)$, i.e., $V_1 + V_2$ is also solvable. Since these subspaces are all in the finite dimensional space C^n, we conclude that a maximal solvable subspace exists. Let V denote the maximal solvable subspace. Thus $AV = BV$ and V contains any other solvable subspace. Now every element $v \in V$ can be the zeroth component for a null vector (i.e., $v = z_0$ and $z_i \in -A_{|V}^{-1}Bz_{i+1}$, $z_{i+1} \in -B_{|V}^{-1}Az_i$). Conversely, every null vector $z \in Z$ has a zero component in V, which, by (2.7), means that all the components are in V. Moreover, if $\dim V = \dim AV\,(=\dim BV)$, then the operators $A_{|V}$ and $B_{|V}$ are one-one and onto the subspace $AV\,(=BV)$. Thus $\dim Z = \dim V$ in this case.

Now suppose that $\dim AV < \dim V$. Let $0 \neq v_1 \in V \cap \mathcal{N}(B)$ and $0 \neq v_2 \in V \cap \mathcal{N}(A)$, where $\mathcal{N}(\cdot)$ denotes *null space*. Then there exists $z \in Z$ of the form

$$z = (\dots, z_{-i-1}, z_{-i}, 0, 0, \dots, 0, z_0, z_1, \dots)^t$$

with $z_0 = v_1$ and $z_{-i} = v_2$. It follows that Z is infinite dimensional, since we can choose i arbitrarily. ■

The above proof is constructive, i.e., it shows that we can find Z by finding the maximal solvable subspace V. (This is done in the next two sections.) Elements $v_1 \in \mathcal{N}(A) \cap V$ and $v_2 \in \mathcal{N}(B) \cap V$ give rise to an infinite dimensional subspace Z, while if $(\mathcal{N}(A) + \mathcal{N}(B)) \cap V = \{0\}$, each nonzero element $v \in V$ yields a unique element $z \in Z$.

In the special case that A and B are square, we get

Corollary 2.1. *Suppose that A and B are square. Then the subspace Z is finite dimensional if and only if*

$$\det(A + \lambda B) \not\equiv 0,$$

i.e., the determinant of the matrix pencil $A + \lambda B$ is not identically 0.

Proof. If $\det(A + \lambda B) \equiv 0$, then for each λ there exists $v_\lambda \neq 0$ such that

$$(A + \lambda B) v_\lambda = 0.$$

Thus $z_\lambda = (\ldots, \lambda^{-1} v_\lambda, v_\lambda, \lambda v_\lambda, \ldots) \in Z$. Thus $\dim Z = \infty$.

Conversely, suppose that $\det(A + \lambda B) \not\equiv 0$. Let us assume that $\dim Z = \infty$. Then by the theorem, we have that $\dim V > \dim AV = \dim BV$. Since $AV = BV$, we get that $\dim V > \dim (A + \lambda B) V$. Thus $A + \lambda B$ is singular for each λ, which contradicts that $\det(A + \lambda B) \not\equiv 0$. ■

Note that if $m < n$, then there exist $0 \neq v_\lambda \in \mathcal{N}(A + \lambda B)$, for each λ. This again yields that $\dim Z$ is infinite. (See Corollary 3.2 for an alternate proof.) If $\dim Z$ is finite, then we have shown that $z = (z_i) \in Z$ implies that $\operatorname{span}\{z_i\} \subset V$. Moreover, each $z \in Z$ can be generated by some $z_0 \in V$.

3. THE KRONECKER CANONICAL FORM

In this section we study the subspace Z by considering the Kronecker canonical form of the matrix pencil $A + \lambda B$. We call the subspace V of R^n *deflating* if

$$\dim(AV + BV) = \dim V.$$

This is a further generalization of the concept of invariant subspaces to matrix pencils. We have seen that the dimension of Z is finite if and only if the maximal solvable subspace (relative to A and B) is deflating.

The generalization of the Jordan canonical form to the matrix pencil $A + \lambda B$ is given by the Kronecker canonical form (see e.g. [5] or [4]),

$$P(A+\lambda B)Q = \operatorname{diag}\left(L_{\varepsilon_1},\dots,L_{\varepsilon_p}, L^t_{\eta_1},\dots,L^t_{\eta_q}, \lambda N - I, \lambda I - J\right),$$

where

(1) L_μ is the $\mu \times (\mu+1)$ bidiagonal pencil $\begin{bmatrix} \lambda & & -1 & & 0 \\ & \ddots & & \ddots & \\ 0 & & \lambda & & -1 \end{bmatrix}$,

(2) L^t_μ is the transpose of L_μ,

$$\begin{bmatrix} \lambda & \ddots & 0 \\ -1 & \ddots & \lambda \\ & \ddots & \\ 0 & & -1 \end{bmatrix},$$

(3) N is a nilpotent Jordan matrix, and

(4) J is in Jordan canonical form.

Thus $\lambda I - J$ contains the finite eigenvalues (i.e., $(A+\lambda B)\chi_\lambda = 0$), and $\lambda N - I$ the infinite eigenvalues (i.e., $Bx = 0$), while L_{ε_i} and $L^t_{\eta_j}$ contain the column and row singularities, i.e., there exist polynomial (column and row) vectors that zero out the pencil identically. The sizes of these blocks are given by the Kronecker column indices ε_i and Kronecker row indices η_j respectively.

THEOREM 3.1. *The dimension of* Z *is finite if and only if the Kronecker canonical form of* $A + \lambda B$ *has no column indices. In this case* dim Z *equals the number* (*counting multiplicity*) *of nonzero, finite eigenvalues of* $A + \lambda B$.

Proof. First suppose that there exists a column index $\varepsilon_1 = \mu$. Then for each scalar λ, $(A+\lambda B)v_\lambda = 0$, where the polynomial vector

$$v_\lambda = Q(0,\dots,0,1,\lambda,\lambda^2,\dots,\lambda^\mu,0,\dots,0)^t. \tag{3.1}$$

Thus, as in the proof of Corollary 2.1, we see that $\dim Z = \infty$.

Now suppose that there are no column indices. Let us find the maximal solvable subspace V. We have the Kronecker form

$$P(A+\lambda B)Q = \begin{bmatrix} E+\lambda F & & & 0 \\ & \lambda J_1 - I & & \\ 0 & & J_2 - \lambda I & \\ & & & J_3 - \lambda I \end{bmatrix}, \tag{3.2}$$

where $E+\lambda F$ contains the (row) singularities, J_1 and J_2 are nilpotent Jordan matrices, and J_3 is a Jordan matrix with no nilpotent blocks, i.e. a nonsingular Jordan matrix.

Since $Ax_k + Bx_{k+1} = 0$ if and only if $PAQQ^{-1}x_k + PBQQ^{-1}x_{k+1} = 0$, we can restrict ourselves to finding the maximal solvable subspace (call it U) relative to

$$C = \begin{bmatrix} E & & & 0 \\ & -I & & \\ & & J_2 & \\ 0 & & & J_3 \end{bmatrix}, \qquad D = \begin{bmatrix} F & & & 0 \\ & J_1 & & \\ & & -I & \\ 0 & & & -I \end{bmatrix}.$$

(Then we get $V = QU$). Since J_1 and J_2 are nilpotent and E and F are composed of blocks of type

$$\begin{bmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & & 0 \\ & \ddots & \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \tag{3.3}$$

respectively, we see that

$$U = \{(0,0,\dots,0,u_t,u_{t+1},\dots,u_n)^t : u_i \text{ arbitrary and } t \text{ the first column of } J_3\}.$$

Moreover, since J_3 and I are both nonsingular and of the same dimensions, we conclude that U is deflating. Thus $V = QU$, the maximal subspace relative to A and B, is also deflating. By Theorem 2.1, $\dim Z = \dim V = \dim U$, which equals the dimension of J_3. ■

We can now improve Corollary 2.1.

Corollary 3.1. *If A and B are square, then the subspace Z is finite dimensional if and only if*

$$\det(A+\lambda B) \not\equiv 0.$$

Furthermore, in this case the precise dimensionality of Z is k where $\det(A+\lambda B) = K\lambda^l \prod_1^k (\lambda - \lambda_i)$ *and* $\lambda_i \neq 0$, $i = 1,\dots,k$.

Proof. The proof is immediate from the theorem, since $\det(A+\lambda B) \not\equiv 0$ if and only if $A+\lambda B$ has no singularities, i.e. if and only if the Kronecker form has no column (or row) indices. ■

COROLLARY 3.2. *If $m < n$, then* $\dim Z$ *is infinite.*

Proof. If $m < n$, then the columns of $A + \lambda B$ are linearly dependent, for every scalar λ. Thus, the Kronecker form has column indices which, by the theorem, yields the conclusion. ∎

If $\dim Z$ is finite, then $A + \lambda B$ has the Kronecker form in (3.2) and, as seen in the proof of Theorem 3.1, the maximal solvable subspace (relative to A and B) is

$$V = QU,$$

where $U = \{u = (u_i) \in R^n : u_i = 0,\ i = 1,2,\dots,t-1\}$, and t is the first column of the non-nilpotent blocks J_3. Each vector $z_0 \in V$ generates an element $z = (z_i) \in Z$, and all of Z is generated in this way.

4. AN ALGORITHM FOR V

We now present a finite iterative algorithm for finding V, the maximal solvable subspace (relative to A and B). We first present some preliminary results. We will use the generalized inverse of an $m \times n$ matrix A. Let $\mathcal{N}(A)$ denote the null space of A and $\mathcal{R}(A)$ the range space of A. We say that A^+ is a *generalized inverse of* A if

$$AA^+A = A.$$

A^+ is also called a $\{1\}$-inverse of A (see [1]).

If we are given the system of linear equations

$$Ax = b, \qquad x \in C^n,$$

then x is a solution if and only if $x = A^+b$ for some generalized inverse A^+. In fact, the set of all solutions [if it is nonempty, i.e. if $b \in \mathcal{R}(A)$] is equal to $A^+b + \mathcal{N}(A)$. Moreover, $P_{\mathcal{R}(A)} = AA^+$ is a projection onto $\mathcal{R}(A)$ and $x = A^+b$ is always a solution of $Ax = P_{\mathcal{R}(A)}b$.

If x is restricted to a subspace W, i.e.

$$Ax = b, \qquad x \in W,$$

then a solution exists if and only if a solution exists for

$$AP_Wx = b,$$

where P_W is some projection onto W. Moreover, a solution is $P_W(AP_W)^+b$, and $P_W(AP_W)^+$ is called a *W-restricted generalized inverse of A*.

Now let

$$\begin{aligned} A_0 &:= A, \qquad B_0 := B, \\ S_i &:= \mathscr{R}(A_i)\cap\mathscr{R}(B_i) \\ U_i &:= A_i^{-1}(S_i)\cap B_i^{-1}(S_i), \end{aligned} \tag{4.1}$$

where A_i^{-1} (and B_i^{-1}) denotes inverse image, i.e.

$$U_i = [A_i^+(S_i)+\mathscr{N}(A_i)]\cap[B_i^+(S_i)+\mathscr{N}(B_i)], \tag{4.2}$$

and

$$A_{i+1} := A_{i|U_i}, \qquad B_{i+1} := B_{i|U_i}. \tag{4.3}$$

LEMMA 4.1. *Let $z=(z_j)\in Z$. Then, for $i=0,1,2,\ldots,$ and $-\infty<j<\infty$, we have*

$$Az_j = A_i z_j, \qquad Bz_j = B_i z_j, \tag{4.4}$$

$$U_{i+1}\subset U_i, \qquad S_{i+1}\subset S_i, \tag{4.5}$$

$$z_j\in U_i. \tag{4.6}$$

Proof. Now

$$\begin{aligned} S_1 &= \mathscr{R}(A_1)\cap\mathscr{R}(B_1) \\ &= \mathscr{R}(A_{|U_0})\cap\mathscr{R}(B_{|U_0})\subset S_0, \\ U_1 &= A_1^{-1}(S_1)\cap B_1^{-1}(S_1) \\ &= A_{|U_0}^{-1}(S_1)\cap B_{|U_0}^{-1}(S_1) \\ &\subset A^{-1}(S_0)\cap B^{-1}(S_0) = U_0. \end{aligned}$$

Moreover, since $Az_j = -Bz_{j-1}$, we see that both Az_j and Bz_j are in S_0, so that $z_j\in U_0$. This proves (4.4) to (4.6). ∎

Since S_0 and U_0 are finite dimensional, the iteration must stop in a finite number of steps, say k, i.e.

$$S_k = S_{k+1}, \qquad U_k = U_{k+1}.$$

THEOREM 4.1. *The subspace U_k found above is the maximum solvable subspace relative to A and B.*

Proof. If V is the maximum solvable subspace, then we have

$$AV = BV \subset S_0.$$

Thus $V \subset U_0$. Similarly, $AV = BV \subset S_i$ and $V \subset U_i$, $i = 1, \dots, k$. Since the iteration stops, we have

$$A_k U_k = AU_k = S_k,$$

$$B_k U_k = BU_k = S_k.$$

Thus since V is the maximal solvable subspace, we must have $V = U_k$. ∎

Now if $\{z_k\} \in Z$, i.e. $Az_k + Bz_{k+1} = 0$, we know that $z_k \in V$ for all k. Let $A_{|V}^{+}$ be any restricted (to V) generalized inverse of A, i.e. $A_{|V}^{+} = P_V(AP_V)^{+}$. Then

$$z_k \in -A_{|V}^{+} B z_{k+1} + \mathscr{N}(A_{|V}).$$

This shows (again) that Z is finite dimensional if and only if V is deflating, i.e. if and only if $\mathscr{N}(A_{|V}) = \{0\}$. [Note that this implies that $\mathscr{N}(B_{|V}) = \{0\}$ also.]

The steps outlined in (4.1) to (4.3) provide an algorithm for finding V. The algorithm restricts A and B to smaller and smaller subspaces U_i until V is found. Note that finding $\mathscr{R}(A) \cap \mathscr{R}(B)$ is equivalent to finding $\mathscr{N}(A, -B)$. For, if one finds, using e.g. Gaussian elimination, $n \times t$ matrices C and D such that

$$\mathscr{N}(A, -B) = \mathscr{R}\begin{pmatrix} C \\ D \end{pmatrix},$$

then

$$z \in S_0 = \mathscr{R}(A) \cap \mathscr{R}(B)$$

if and only if

$$z = Ax = By$$

for some x and y; i.e. if and only if

$$Ax - By = [A, -B]\begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Therefore,

$$S_0 = \mathscr{R}(AC) = \mathscr{R}(BD).$$

Now $u \in U_0$ if and only if

$$u = a + n_a = b + n_b$$

with $n_a \in \mathscr{N}(A)$, $n_b \in \mathscr{N}(B)$, and both $Aa, Bb \in S_0$. Thus $Aa = z_a = By_a$ and $Bb = z_b = Ay_b$, for some y_a and y_b. But then

$$[A, -B]\begin{pmatrix} a \\ y_a \end{pmatrix} = [A, -B]\begin{pmatrix} y_b \\ b \end{pmatrix} = 0,$$

so that $a \in \mathscr{R}(C)$ and $b \in \mathscr{R}(D)$. Now since $n_a \in \mathscr{R}(C)$ and $n_b \in \mathscr{R}(D)$, we get that

$$U_0 = \mathscr{R}(C) \cap \mathscr{R}(D).$$

Let E be an $n \times r$ full column rank matrix such that

$$\mathscr{R}(E) = \mathscr{R}(C) \cap \mathscr{R}(D) = U_0$$

[E can be found using $\mathscr{N}(C, -D)$ just as S_0 was found above]. Then

$$A_1 = AE \quad \text{and} \quad B_1 = BE.$$

One now begins again with A_1 replacing A and B_1 replacing B. In summary, there are two basic operations at each step. First, find C and D such that

$$\mathscr{N}(A, -B) = \begin{bmatrix} C \\ D \end{bmatrix}.$$

Then find E such that

$$\mathcal{R}(E) = \mathcal{R}(C) \cap \mathcal{R}(D),$$

and replace A by AE and B by BE. We stop when E is the identity matrix. The maximal solvable subspace V is the range of the product of the matrices E found.

5. CONCLUSION

In this paper we have studied the kernel Z of the biinfinite block Toeplitz matrix

$$D = \begin{bmatrix} \ddots & \ddots & & & \\ & A & B & & \\ & & A & B & \\ & & & \ddots & \ddots \end{bmatrix}.$$

It was shown that $\dim Z < \infty$ if and only if the matrix pencil $A + \lambda B$ has no column singularities—i.e., there are no Kronecker column indices in the Kronecker canonical form, or equivalently, there is no polynomial vector $q(\lambda) = \lambda^p v_p + \lambda^{p-1} v_{p-1} + \cdots + v_0$ such that $(A + \lambda B)q(\lambda) \equiv 0$. In this case, $\dim Z$ equals the number (counting multiplicity) of nonzero finite eigenvalues of the pencil $A + \lambda B$. This was shown using V, the maximal solvable subspace (relative to A and B). In fact, V is deflating if and only if $\dim Z < \infty$, and in this case $\dim Z = \dim V$.

Now D is one-one (on the sequence space) if and only if $\dim Z = 0$, if and only if $\dim V = 0$, if and only if $A + \lambda B$ has no column singularities and no nonzero finite eigenvalues. Similarly, D is onto if and only if $\dim \mathcal{N}(D^*) = 0$, if and only if $A + \lambda B$ has no row indices and no nonzero finite eigenvalues. Since $m < n$ $(m > n)$ implies that $A + \lambda B$ has column (row) indices, we conclude that D is one-one and onto if and only if we have (1) A and B are square, (2) $A + \lambda B$ has no singularities, and also (3) $A + \lambda B$ has no nonzero eigenvalues.

If A is nonsingular and $A + \lambda B = 0$ for some scalar $\lambda = \alpha \neq 0$, then $\dim Z = n$, since α is a nonzero eigenvalue with n linearly independent eigenvectors, i.e., the Kronecker form is $-\alpha I + \lambda I$. Note that $\dim Z = n$ if and only if $A + \lambda B$ has no column singularities and no nonzero eigenvalues, i.e. if and only if the Kronecker form is $\lambda I + J$ where J is an $n \times n$ Jordan matrix with no nilpotent blocks (since necessarily $m = n$).

If either A or B is nonsingular, then necessarily $\dim Z < \infty$, since $A + \lambda B$ cannot have any singularities. In fact, $\det(A + \lambda B) \neq 0$ for $\lambda = 0$ (for $|\lambda|$ large) if A (B) is nonsingular.

In the case that $\dim Z < \infty$, a basis for Z can be found using the maximal solvable subspace V. In fact,

$$Z = \{z = (\dots, z_{-1}, z_0, z_1, \dots) : z_0 \in V, z_i = -A_{|V}^{+} B z_{i+1}\},$$

where $A_{|V}^{+}$ is any (restricted to v) generalized inverse of A. Note that $A_{|V}^{+} B : V \to V$ is one-one and onto.

The maximal solvable subspace V can be found using the (finite) algorithm of Section 4. Also, V corresponds to the span of the generalized eigenvectors of all grades corresponding to the nonzero (finite) eigenvalues.

The numerical calculation of the Kronecker canonical form might prove unstable (see e.g. [3]) due to the possible ill conditioning of the transformations P and Q. One can instead use unitary transformations U_1 and U_2 to reduce the pencil $A + \lambda B$ to the form

$$U_1(A + \lambda B)U_2 = \begin{bmatrix} A_l + \lambda B_l & 0 & 0 & 0 \\ * & A_f + \lambda B_f & 0 & 0 \\ * & & A_i + \lambda B_i & 0 \\ * & * & * & A_r + \lambda B_r \end{bmatrix}$$

where (1) $A_l + \lambda B_l$ and $A_r + \lambda B_r$ contain the row and column (left and right) singularities, (2) $A_i + \lambda B_i$ contains the infinite eigenvalues, and (3) $A_f + \lambda B_f$ contains the finite eigenvalues. The above form can be obtained using a backward stable algorithm (see e.g. [4]). Thus, $\dim Z < \infty$ if and only if there is no block $\lambda B_r - A_r$, and in this case $\dim Z$ equals the number of nonzero eigenvalues in $A_f + \lambda B_f$. A basis for Z can be obtained from the corresponding columns of U_2.

The numerical stability of finding $\dim Z$ (when $\dim Z < \infty$) and that of finding a basis for Z are, in a certain sense, reversed. This is because $\dim Z < \infty$ when there are no column singularities, and $\dim Z$ corresponds to the *nonzero* eigenvalues of $A + \lambda B$ (whereas usually the null space corresponds to finding the zero eigenvalues, which is an unstable process). To find $\dim \mathcal{N}(D)$ of a matrix D is unstable, except in the full rank situation. In fact, the rank of a matrix is an upper-semicontinuous function and *not* a continuous one. Thus, small perturbations of the elements of a matrix D can increase the rank from deficient rank to higher or even to full rank, but not vice versa. Equivalently, small perturbations can reduce the dimension of the null space but not increase it. However, since dim Z corresponds to the nonzero eigenvalues of $A+\lambda B$, the situation is reversed. Small perturbations (in A and B) can increase dim Z but not decrease it. So the value of dim Z is completely stable when dim $Z=n$. (For a matrix D, $m\times n$, the value of dim $\mathscr{N}(D)$ is stable when it is $n-m$, i.e. when the matrix D is full rank).

We now find dim Z for the following examples, using the algorithm in Section 4.

EXAMPLE 5.1. Let $A=I$ and

$$B=\begin{bmatrix}1&1&0\\0&1&0\\0&0&0\end{bmatrix}.$$

Then

step 1:
$$\mathscr{N}(A,-B)=\mathscr{R}\begin{pmatrix}C\\D\end{pmatrix}=\mathscr{R}\begin{bmatrix}1&1&0\\0&1&0\\0&0&0\\1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$U_0=\mathscr{R}(C)\cap\mathscr{R}(D)=\mathscr{R}(C),\qquad E=C.$$

Then

$$A_1=AE=E,\qquad B_1=BE=\begin{bmatrix}1&2&0\\0&1&0\\0&0&0\end{bmatrix}.$$

step 2:
$$\mathscr{N}(A_1,-B_1)=\mathscr{N}\begin{bmatrix}1&0&0&-1&-1&0\\0&1&0&0&-1&0\\0&0&0&0&0&0\end{bmatrix}$$

$$=\mathscr{R}\begin{pmatrix}C\\D\end{pmatrix}=\mathscr{R}\begin{bmatrix}0&1&1&0\\0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$

$$U_1=\mathscr{R}(C)\cap\mathscr{R}(D)=R^3,\qquad E=I.$$

So we stop and conclude that

$$V=\mathscr{R}\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}.$$

Since $S_1=\mathscr{R}(A_1)\cap\mathscr{R}(B_1)=V$, we get that $\dim Z=\dim V=2$.

EXAMPLE 5.2. Let

$$A=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\end{bmatrix},\qquad B=\begin{bmatrix}3&1&5\\5&2&9\\7&1&9\end{bmatrix}.$$

Then

step 1: $$(A,B)\sim\begin{bmatrix}1&0&1&2&0&2\\0&1&1&1&0&1\\0&0&0&0&1&2\end{bmatrix},$$

so

$$\mathscr{N}(A,B)=\mathscr{R}\begin{pmatrix}C\\D\end{pmatrix}=\mathscr{R}\begin{bmatrix}2&1&2\\1&1&1\\0&-1&0\\-1&0&0\\0&0&0\\0&0&-1\end{bmatrix},$$

$$(C,-D)\sim\begin{bmatrix}1&0&0&1&0&1\\0&1&0&0&0&1\\0&0&1&0&0&3\end{bmatrix},$$

$$\mathscr{N}(C,-D)=\mathscr{R}\begin{pmatrix}F\\G\end{pmatrix}=\mathscr{R}\begin{bmatrix}1&0&1\\0&0&1\\-1&0&0\\0&0&3\\0&-1&0\\0&0&-1\end{bmatrix},$$

$$U_0=\mathscr{R}(C)\cap\mathscr{R}(D)$$
$$=\mathscr{R}(CF)=\mathscr{R}\begin{bmatrix}3\\2\\-1\end{bmatrix},\qquad E=\begin{bmatrix}3\\2\\-1\end{bmatrix},$$

$$A_1=AE=\begin{bmatrix}3\\5\\7\end{bmatrix},\qquad B_1=BE=\begin{bmatrix}6\\10\\14\end{bmatrix}.$$

Since $B_1 = 2A_1$, we stop. We get that 2 is a generalized eigenvalue and $\dim Z = 1$. Note that $V = \mathcal{R}(E)$. In fact, the biinfinite vectors in Z are generated by multiples of

$$z_0 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \quad \text{with} \quad z_j = 2z_{j+1}.$$

EXAMPLE 5.3. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 1 & 3 \end{bmatrix}.$$

Since the vector

$$v = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

is in $\mathcal{N}(A) \cap \mathcal{N}(B)$, we conclude immediately that dim Z is infinite. In fact, we can choose $z = (z_i) \in Z$ with $z_i = f(i)v$, where $f(i)$ is any real valued function.

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Received 23 June 1982; revised 9 February 1983