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Zero Duality Gaps in Infinite-Dimensional Programming[1,2]

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Abstract. In this paper we study the following infinite-dimensional programming problem:

(P) $\mu := \inf f_0(x)$, subject to $x \in C, f_i(x) \leq 0, i \in I$,

where I is an index set with possibly infinite cardinality and C is an infinite-dimensional set. Zero duality gap results are presented under suitable regularity hypotheses for convex-like (nonconvex) and convex infinitely constrained program (P). Various properties of the value function of the convex-like program and its connections to the regularity hypotheses are studied. Relationships between the zero duality gap property, semicontinuity, and ϵ-subdifferentiability of the value function are examined. In particular, a characterization for a zero duality gap is given, using the ϵ-subdifferential of the value function without convexity.

Key Words. Zero duality gaps, convex-like infinite programs, value function, semi-infinite programming, subdifferentiability.

1. Introduction

In this paper, we study the following optimization problem:

$$\text{(P)} \quad \mu := \inf f_0(x),$$
$$\text{subject to } x \in C, \quad f_i(x) \leq 0, \quad i \in I,$$

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where X is a separated, locally convex, topological vector space, C is a nonempty subset of X, I is an index set with cardinality (card I) possibly infinite, and $f_i: C \to \mathbf{R}$, $i \in \bar{I} := I \cup \{0\}$. We let $Y := \prod_I \mathbf{R}$ denote the product space in the product topology and Y^* the continuous dual space of Y with the convex core topology; Y^* is the generalized finite sequence space consisting of all functionals $f: I \to \mathbf{R}$ with finite support; see Charnes, Cooper, and Kortanek (Ref. 1).

For $\lambda \in Y^*$,

$$\phi(\lambda) := \inf_{x \in C} L(x, \lambda)$$

denotes the dual functional of (P), where

$$L(x, \lambda) := f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \tag{1}$$

is the Lagrangian function of (P). Then, the Lagrangian dual of (P) is

$$\text{(D)} \quad \nu := \sup_{\lambda \in \Lambda} \phi(\lambda),$$

where the dual cone

$$\Lambda := \{\lambda = (\lambda_i) \in Y^*: \lambda_i \geq 0, i \in I, \text{ and } \lambda_i = 0 \text{ for all but a finite number of } i\}.$$

The value $\mu - \nu$ is called the duality gap for (P); the term defect has also been used (Ref. 2). Weak duality implies that the gap is nonnegative. If there exists a Lagrange multiplier vector $\lambda \in \Lambda$ such that $\mu = \nu = \phi(\lambda)$, then the duality gap is zero and strong duality holds. However, it is well known that a Lagrange multiplier vector may not exist; see, e.g., Example 1 of Duffin and Karlovitz (Ref. 3), where $\mu = 0 > \nu = -\infty$ is a nonzero duality gap. On the other hand, a Lagrange multiplier vector may not exist while the duality gap is zero, e.g.,

$$\begin{aligned} 0 = \mu &= \inf\{-x: x^2 \leq 0\}, \\ 0 = \nu &= \sup_{\lambda \geq 0} \inf_x -x + \lambda x^2 = \lim_{\lambda \to \infty} \inf_x -x + \lambda x^2. \end{aligned} \tag{2}$$

Thus, the regularity conditions that guarantee a zero duality gap can be significantly weaker than the regularity conditions needed for the existence of a Lagrange multiplier vector.

It is well known that the existence of a Lagrange multiplier is equivalent to stability of the program. Let $V(u)$ be the value function of (P), i.e., the objective value subject to perturbations of the right-hand side. Then, for convex programs, the existence of a Lagrange multiplier is equivalent to the subdifferential of the value function being nonempty (e.g., see Refs. 4 and 5). This guarantees that the value function $V(u)$ is bounded below by an affine function $h(u)$, where $V(0)=\mu=h(0)$, and that V cannot decrease too rapidly at $u=0$. However, no Lagrange multiplier may exist, and we can still have a well-posed program; i.e., we can still have a zero duality gap, primal attainment, and lower semicontinuity of the optimum value with respect to perturbations of the right-hand side. Moreover, the value function can still be bounded below by affine functions $h_\epsilon(u)$, where now $h_\epsilon(0)=\mu-\epsilon$, $\epsilon>0$ arbitrary. See Sections 3 and 4.

Further, many algorithms solve (P) by working with the dual problem (D), and for many problems it is known that, as one solves a perturbed problem of (P), say (P_ϵ), where the perturbations are a result of, e.g., numerical approximations in the data, then the corresponding optimal values $\mu(\epsilon)\to\nu$ as the perturbations $\epsilon\to 0^+$. This is also true if one discretizes or solves finite approximations of (P) formed by using only a finite number of the constraints (e.g., see Refs. 3, 6, and 7). Thus, from theoretical and computational points of view, a zero duality gap $\mu=\nu$ is important. In fact, a central issue in infinite-dimensional programming has been the identification of sufficient conditions for zero duality gap.

The model program (P) includes a large class of problems. Duality gap results have been studied for special classes of (P). For example, if (P) is a semi-infinite linear program (i.e., the functions f_i, $i\in\tilde{I}$, are linear and $C=\mathbf{R}^n$), then conditions that guarantee a zero duality gap have been given in Ref. 2. Conditions for semi-infinite convex programs, where f_i, $i\in\tilde{I}$, and $C\subset\mathbf{R}^n$ are convex, have been obtained, in terms of recession cone conditions in Ref. 8 and by using certain constraint regularity conditions in Ref. 9. Asymptotic duality results for semi-infinite programming problems are given in Refs. 10 and 11. Duality results which use modified dual programs are also known in the semi-infinite case (Refs. 12, 13, 14). A unified approach using conjugate duality is given in Ref. 5.

Most conditions need the convexity of the functions and the finite-dimensional structure of the set C. The purpose of this paper is to examine zero duality gap results for (P) without these restrictions. In Sections 2, 3, 4 of this paper, the convexity hypotheses are substantially weakened and the set C is not restricted to be finite dimensional. Throughout this paper, the set Π introduced in Section 2 plays an important role in conditions for a zero duality gap and conditions for attainment of the optimum.

We now sketch an outline of the paper. In Section 2, we introduce our notions of regularity and strong regularity in terms of a set Π in a product space, and we present a zero duality gap result for regular convex-like programs (P); see (3). These convex-like programs provide a class of programs which are not convex, but possess many of the nice properties that convex programs have. In fact, we get some of the results which can be derived from conjugate duality theory as in Ref. 5. Sufficient conditions for the regularity condition are given. In Section 3, we examine various properties of the value function of (P) and relate the regularity condition in terms of the lower semicontinuity of the value function. Section 4 characterizes a zero duality gap for (P) using an ϵ-subdifferential of the value function, but without convexity assumptions. Section 5 presents zero duality gap results for convex and generalized linear programs as consequences of results in Section 2.

2. Convex-Like Programs

The program (P) in Section 1 is called convex-like if, for all $x_1, x_2 \in C$, there exist $x_3 \in C$ such that

$$f_i(x_3) \leq k f_i(x_1) + k f_i(x_2), \qquad i \in \tilde{I}. \tag{3}$$

Here and elsewhere in the paper, $k = 1/2$. We note that the condition (3) is a weaker form of the concept of convex-like functions introduced by Fan in Ref. 15, which we call F-convex-like and define as follows: for all $x_1, x_2 \in C$, and $\alpha \in (0, 1)$, there exist $x_3 \in C$ such that

$$f_i(x_3) \leq \alpha f_i(x_1) + (1-\alpha) f_i(x_2), \qquad i \in \tilde{I}. \tag{4}$$

Different versions of convex-like cone constrained minimization problems were examined in Refs. 16, 17, 18, where Slater-type constraint qualifications are used.

The convex-like programs (P) share an important property, i.e., that the closure of the set Π defined in Proposition 2.1 below is convex, and moreover this closure equals the closure of the epigraph of the value function of (P). These problems provide a class of nonconvex problems which possess many of the nice properties of convex problems. Note that, by convex problems, we mean those problems for which C is convex and f_i is convex, $i \in \tilde{I}$. We could have used the property that the closure of Π is convex as in our definition of convex-like programs, but we keep the above definition for concreteness.

Problem (P) is convex-like if C is a midpoint convex set and each f_i, $i \in \tilde{I}$, is midpoint convex (Ref. 19). If $(X, \leq)$ is a partially ordered space with the partial order $\leq$, C is a totally ordered subset of X; and if, for each $i \in \tilde{I}$, f_i is isotonic [that is, $x \leq y \Rightarrow f_i(x) \leq f_i(y)$], then (P) is convex-like. In fact, if $x_1 \leq x_2$, then we can choose $x_3 := x_1$ in (3).

Let $C \subset X$ and $h: C \to \mathbf{R}$. Let Ω be a midpoint convex set containing $h(C)$; and let $g_i: \Omega \to \mathbf{R}$, for $i \in \tilde{I}$, be a midpoint convex isotonic function. Define, for each $i \in \tilde{I}$, the composite function $f_i: C \to \mathbf{R}$ as

$$f_i(x) = (g_i \circ h)(x) = g_i(h(x)), \qquad x \in C.$$

Then, (P) is a convex-like program.

The following is a concrete example of a nonconvex convex-like program. Let

$$C = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 > 1\}$$
$$\cup \{(1, 0), (0, 1), (0, 0)\},$$

and let $\tilde{I} = N$, the set of all positive integers. Define $f_i: C \to \mathbf{R}$ by

$$f_i(x_1, x_2) = x_1/i + [(i-1)/i]x_2, \qquad (x_1, x_2) \in C. \tag{5}$$

Note that the set C is not convex [consider the points $(1, 0)$ and $(0, 1)$]. For related definitions and examples, see Refs. 16 and 17.

The next proposition provides an interesting characterization for an infinite convex-like program in terms of convexity of a set. Define $F: X \to \mathbf{R}^I$ by

$$F(x) = (f_i(x))_{i \in I}. \tag{6}$$

Proposition 2.1. Program (P) is convex-like [respectively, F-convex-like] if and only if the set

$$\Pi = (F, f_0)(C) + \mathbf{R}_+^I \times \mathbf{R}_+$$
$$= \{(u, r) \in \mathbf{R}^I \times \mathbf{R}: \exists x \in C, f_0(x) \leq r, \text{ and } f_i(x) \leq u_i, i \in I\}$$

is midpoint convex [resp., convex].

Proof. Necessity follows immediately from the definitions of convex-like programs. To prove sufficiency, let $x_1, x_2 \in C$. Then,

$$((f_i(x_1))_{i \in I}, f_0(x_1)), ((f_i(x_2))_{i \in I}, f_0(x_2)) \in \Pi.$$

Since Π is midpoint convex,

$$((kf_i(x_1) + kf_i(x_2))_{i \in I}, kf_0(x_1) + kf_0(x_1)) \in \Pi.$$

Then, there exists $x_3 \in C$ such that

$$f_i(x_3) \leq kf_i(x_1) + kf_i(x_2), \qquad i \in \tilde{I},$$

and hence (P) is convex-like. The proof for F-convexlike is similar. □

We call a program consistent if it has a feasible point, and we call it regular if μ is finite and there exist a neighborhood U of 0 in Y and a constant $\gamma > \mu$ such that

$$\Omega = \Pi \cap \bar{U} \times (-\infty, \gamma] \tag{7}$$

is closed in $Y \times \mathbf{R}$. The program (P) is said to be strongly regular if Π is closed in Y. The set Π is the tool that we use to guarantee zero duality gap. Clearly, the strong regularity condition implies the regularity condition of (P). The following elementary property, used to study minimax theorems in Borwein and Jeyakumar (Ref. 18), will be useful in the proof of our main theorem in this section.

Lemma 2.1. Let $A \subset X$. If A is midpoint convex, then $\bar{A}$ is convex.

Proof. Since the closure of a midpoint convex set is midpoint convex, let us assume that A is closed. Let $x, y \in A$ and let

$$z = \lambda x + (1 - \lambda) y, \qquad \text{for some } 0 < \lambda < 1. \tag{8}$$

We need to show that $z \in A$. Without loss of generality, assume that $\lambda > k = 1/2$. Set

$$z = kx + ky,$$

and then rename $kx + ky$ as y. This gives z in terms of x, y and a new λ in (8). We can continue this process and rename the midpoint $kx + ky$ as y, if $\lambda > k = 1/2$, or as x, if $\lambda < k = 1/2$. We get a possibly finite sequence $\{z^k\}$ converging to z; so, $z \in A$. □

We now present the main result of this section. This theorem provides a sufficient regularity condition to guarantee a zero duality gap. We provide a direct proof using a hyperplane separation argument. In Section 4, we relate this result with the value function, the attainment in (P), the conjugate duality, and the ϵ-subdifferential.

Theorem 2.1. If (P) is a regular convex-like program, then there is zero duality gap for (P) and (D).

Proof. Since (P) has a finite value μ, we see that $(0, \mu - \theta) \notin \bar{\Pi}$, for any fixed real constant $\theta > 0$. Otherwise, there exists a net $\{(y^{\iota}, r^{\iota})\} \subset \Pi$ such that $y^{\iota} \to 0$ and $r^{\iota} \to \mu - \theta$. Since $\theta > 0$, we can choose a subnet $\{(y^{\alpha}, r^{\alpha})\}$ such that

$$(y^{\alpha}, r^{\alpha}) \in \Pi \cap \bar{U} \times (-\infty, \gamma] = \Omega;$$

see (7). From the regularity hypothesis,

$$\lim_{\alpha}(y^{\alpha}, r^{\alpha}) = (0, \mu - \theta) \in \Omega.$$

Then, there exists $x_0 \in C$ such that

$$f_i(x_0) \leq 0, \; i \in I, \quad \text{and} \quad f_0(x_0) \leq \mu - \theta.$$

So,

$$\mu \leq f_0(x_0) \leq \mu - \theta,$$

a contradiction. Therefore, by Lemma 2.1, the set $\Gamma_\theta = \bar{\Pi} + \{(0, \theta - \mu)\}$ is a nonempty closed convex set and moreover $\{0\} \cap \Gamma_\theta = \varnothing$, since Π is midpoint convex and $(0, \theta - \mu + f_0(\bar{x})) \in \Gamma_\theta$, where $\bar{x}$ is any feasible point for (P).

Now, from the strong separation theorem (e.g., Ref. 19), there exists a nonzero $(\lambda, \lambda_0) \in \mathbf{R}^{I^*} \times \mathbf{R}$ such that, for each $(y, r) \in \Gamma_\theta$,

$$\langle \lambda, y \rangle + \lambda_0 r > 0.$$

Thus,

$$\lambda_0(f_0(x) - (\mu - \theta)) + \sum_{i \in I} \lambda_i f_i(x) > 0, \qquad x \in C.$$

Fix $x_0 \in C$. Let $z \in \mathbf{R}^I_+$. Then, for each $\delta > 0$,

$$\lambda_0(f_0(x_0) - (\mu - \theta)) + \delta \langle \lambda, z \rangle + \sum_{i \in I} \lambda_i f_i(x_0) > 0.$$

Letting $\delta \to \infty$, we get $\lambda \in \Lambda$, the nonnegative dual cone. Similarly, for each $\epsilon > 0$,

$$\lambda_0(f_0(x_0) - (\mu - \theta) + \epsilon) + \sum_{i \in I} \lambda_i f_i(x_0) > 0.$$

Letting $\epsilon \to \infty$, we get $\lambda_0 \geq 0$; i.e., we have shown that $0 \neq (\lambda_0, \lambda) \in \mathbf{R}_+ \times \Lambda$.

Suppose that $\lambda_0 = 0$. Then,

$$\lambda \neq 0 \quad \text{and} \quad \sum_{i \in I} \lambda_i f_i(x) > 0, \qquad x \in C.$$

This contradicts the consistency assumption that there exists $x \in C$ such that $f_i(x) \leq 0$, $i \in I$. So, without loss of generality, we can assume that $\lambda_0 = 1$ and get that

$$f_0(x) + \sum_{i \in I} \lambda_i f_i(x) > \mu - \theta, \qquad x \in C.$$

Since $\theta > 0$ is arbitrary, it follows that

$$\sup_{\lambda \in \Lambda} \inf_{x \in C} f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \geq \mu.$$

The proof is completed by noting that the reverse inequality always holds. □

It is clear that the program (P) is regular if C is compact and, for each $i \in \bar{I}$, $f_i(\cdot)$ is lower semicontinuous. However, our interest here is to find conditions for regularity with noncompact C. The following propositions give us such results.

Proposition 2.2. For the program (P), assume that, for each $i \in \bar{I}$, $f_i(\cdot)$ is lower semicontinuous. If there exists a finite index set $I_0 \subset I$, such that the set

$$\{x \in C: f_0(x) \leq \gamma \text{ and } f_i(x) \leq \epsilon,\ i \in I_0\}$$

is compact, for some $\epsilon > 0$ and $\gamma > \mu$, then (P) is regular.

Proof. Define a neighborhood of 0 in Y by

$$U_0 = (-\epsilon, \epsilon) \times \cdots \times (-\epsilon, \epsilon) \times \prod_{i \in I \setminus I_0} \mathbf{R}.$$

Then, we shall show that the set

$$\Omega_0 = \Pi \cap \bar{U}_0 \times (-\infty, \gamma]$$

is closed in $Y \times \mathbf{R}$. Let $\{(y^\alpha, r^\alpha)\}$ be a net in Ω_0 which converges to $(y, r) \in Y \times \mathbf{R}$. Then, there exists a net $\{x^\alpha\} \subset C$ such that

$$f_0(x^\alpha) \leq r^\alpha, \quad f_i(x^\alpha) \leq y_i^\alpha, \qquad i \in I,$$

$$y_j^\alpha \leq \epsilon, \qquad j \in I_0, \qquad r^\alpha \leq \gamma.$$

Hence,

$$\{x^\alpha\} \subset \{x \in C: f_0(x) \leq \gamma, f_i(x) \leq \epsilon, i \in I_0\},$$

which is compact. Then, choose a subnet $\{x^\beta\} \to \tilde{x} \in C$. Since the functions f_i, $i \in \bar{I}$ are lower semicontinuous,

$$f_0(\tilde{x}) \leq r \quad \text{and} \quad f_i(\tilde{x}) \leq y_i, \qquad i \in I.$$

Thus, $(y, r) \in \Omega_0$, since $y_j \leq \epsilon$, $j \in I_0$, and $r \leq \gamma$. □

Remark 2.1. We note from Proposition 2.2, that (P) is regular if C is closed, f_i, $i \in \bar{I}$, are lower semicontinuous, and the set $\{x \in C: f_i(x) \leq \epsilon, i \in I_0\}$ is compact, for some $\epsilon > 0$, and some finite subset I_0 of I. This holds, since

$$\{x \in C: f_0(x) \leq \gamma, f_i(x) \leq \epsilon, i \in I_0\}$$

is a closed subset of $\{x \in C: f_i(x) \leq \epsilon, i \in I_0\}$.

We call a set $C \subset X$ weakly closed if it is closed in the weak topology of X. Similarly, we say that a real function f is weakly lower semicontinuous if it is lower semicontinuous in the weak topology of X.

Proposition 2.3. Assume that X is a reflexive Banach space and C is a weakly closed subset of X. If, for each $i \in \bar{I}$, $f_i(\cdot)$ is weakly lower semicontinuous on C and if there exists $j \in \bar{I}$ such that

$$\liminf_{x \in C, \|x\| \to \infty} f_j(x) = \infty,$$

i.e., f_j is coercive, then (P) is strongly regular.

Proof. Let $\{(u^t, r^t): t \in T\}$ be a net in Π which converges to $(u, r) \in \mathbf{R}^I \times \mathbf{R}$. Then, there exists a net $\{x^t\} \subset C$ such that

$$f_0(x^t) \leq r^t \quad \text{and} \quad f_i(x^t) \leq u_i^t, \qquad i \in I.$$

Assume that $\{x^t\}$ is bounded. Since C is a weakly closed subset of the reflexive Banach space X, we can choose a subnet $\{x^\gamma\}$ which converges weakly to some $x_0 \in C$. From the weak lower semicontinuity hypothesis on f_i, $i \in \bar{I}$, we get

$$f_0(x_0) \leq r, \qquad f_i(x_0) \leq u_i, \qquad i \in I,$$

and hence $(u, r) \in \Pi$.

Suppose that $\{x^t\}$ is unbounded. Then, we can choose a subnet $\{x^\beta\}$ such that

$$x^\beta \neq 0 \quad \text{and} \quad \|x^\beta\| \to \infty.$$

Hence,

$$\liminf_{\beta \to \infty} f_0(x^\beta) \leq r \quad \text{and} \quad \liminf_{\beta \to \infty} f_i(x^\beta) \leq u_i, \qquad i \in I.$$

This contradicts our assumption. □

Remark 2.2. The above proposition can be applied to several classes of infinite interpolation problems, where objective functions are often norms. Since norms are coercive functions, these problems have a zero duality gap.

We shall show in Section 5 how our regularity conditions can be related to general recession cone conditions used to study semi-infinite programming problems. In fact, in Section 5, a generalization of the Clark-Duffin theorem of Karney (Theorem 2, Ref. 20) is given.

3. Value Function and Regularity

The value function (Ref. 21) $V: \mathbf{R}^I \to [-\infty, \infty]$ for the program (P) is defined by

$$V(z) := \inf\{f_0(x) \mid x \in C, f_i(x) \leq z_i, i \in I\}.$$

Note that the infimum over the empty set is $+\infty$ and that $V(0) = \mu$.

In this section, we present various properties of the value function and its relation to our definition of regularity. In particular, we show the relationships between semicontinuity of V, closure of Π, and attainment in (P).

Proposition 3.1. If the program (P) is convex-like [resp., F-convex-like], then the value function is midpoint convex [resp., convex].

Proof. Suppose that (P) is convex-like. Let $z^1, z^2 \in \mathbf{R}^l$ and $\epsilon > 0$. Then, there exist $x_1, x_2 \in C$, with

$$f_i(x_1) \le z_i^1 \quad \text{and} \quad f_i(x_2) \le z_i^2, \qquad \text{for all } i \in I,$$

$$f_0(x_1) < V(z^1) + \epsilon \quad \text{and} \quad f_0(x_2) < V(z^2) + \epsilon.$$

Then, since (P) is convex-like, there exists $x_3 \in C$ such that, with $k = 1/2$,

$$f_i(x_3) \le kf_i(x_1) + kf_i(x_2) \le kz_i^1 + kz_i^2,$$

$$f_0(x_3) \le kf_0(x_1) + kf_0(x_2) < kV(z^1) + kV(z^2) + \epsilon.$$

Hence,

$$V(kz^1 + kz^2) \le f_0(x_3) < kV(z^1) + kV(z^2) + \epsilon,$$

and so

$$V(kz^1 + kz^2) \le kV(z^1) + kV(z^2),$$

since $\epsilon > 0$ is arbitrary. The proof for the F-convex-like case is similar and is left for the reader. □

Proposition 3.2. If the program (P) is convex-like [resp., F-convex-like], the functions f_i, $i \in \bar{I}$, are positively homogeneous, and C is a cone, then the value function V is midpoint convex [resp., convex] and positively homogeneous.

Proof. Let (P) be convex-like. If we show that V is positively homogeneous, then the conclusion would follow from Proposition 3.1. From the positive homogeneity of f_i, $i \in \bar{I}$, we get, for all $\alpha > 0$,

$$\begin{aligned} V(\alpha z) &= \inf\{f_0(x) \mid x \in C, f_i(x) \le \alpha z_i, i \in I\} \\ &= \inf\{f_0(\alpha x) \mid \alpha x \in C, f(\alpha x) \le \alpha z_i, i \in I\} \\ &= \alpha \inf\{f_0(x) \mid x \in C, f_i(x) \le z_i, i \in I\} \\ &= \alpha V(z). \end{aligned}$$

□

Proposition 3.3. If the program (P) is regular, then the value function V is lower semicontinuous at 0 and (P) attains its minimum.

Proof. Suppose that V is not lower semicontinuous at 0. Then, there exists a net $\{z_\alpha\}$ in $\mathbf{R}^l$ with $z_\alpha \to 0$ such that $V(z_\alpha) \to \bar{V} < V(0) = \mu$. Since (P) is regular, there exists a neighborhood U of 0 and $\gamma > V(0)$ such that $\Omega = \Pi \cap \bar{U} \times (-\infty, \gamma]$ is closed. Now, choose a subnet $\{z_\delta\}$ such that, with $k = 1/2$,

$$(z_\delta, V(z_\delta) + (k[V(0) - \bar{V}])), \quad \text{in } \Omega.$$

Since Ω is closed,

$$\lim_\delta (z_\delta, V(z_\delta) + k[V(0) - \bar{V}]) = (0, k[V(0) + \bar{V}]) \in \Omega.$$

Hence, there exists $x_0 \in C$ such that

$$f_i(x_0) \leq 0, \; i \in I, \quad \text{and} \quad f_0(x_0) \leq k[V(0) + \bar{V}],$$

and so

$$V(0) \leq f_0(x_0) \leq k[V(0) + \bar{V}].$$

This is a contradiction.

We shall now show that (P) attains its minimum. For each positive integer n, the sequence $\{(0, V(0) + 1/n)\} \subset \Pi$. Since $V(0) < \gamma$, we can choose a subsequence $\{(0, V(0) + 1/m)\}$, in Ω. From the closedness of Ω, $(0, V(0)) \in \Omega$. Hence, there exists $\bar{x} \in C$ with $f_i(\bar{x}) \leq 0$, $i \in I$, and $f_0(\bar{x}) \leq V(0)$. Thus, $V(0) = f_0(\bar{x})$. □

We recall that the epigraph of the value function V is given by

$$\text{epi } V = \{(y, r) \in \mathbf{R}^l \times \mathbf{R} \mid V(y) \leq r\}.$$

It is worth noting that epi V and the set Π, used in the definition of regularity, satisfy $\Pi \subset \text{epi } V \subset \bar{\Pi}$. This fact will be used in the proof of the following theorem and proposition. To see this, let $(y, r) \in \text{epi } V$. Then, $V(y) \leq r$. If $V(y) < r$, then there exists $x \in C$ with $f_i(x) \leq y_i$, $i \in I$, and $r > f_0(x)$, and so $(y, r) \in \bar{\Pi}$. If $V(y) = r$, then there exists a net $\{x_\alpha\} \subset C$, $f_i(x_\alpha) \leq y_i$, $i \in I$, and $0 \leq \epsilon_\alpha = f_0(x_\alpha) - r \to 0$, as $\alpha \to \infty$. Then, the net $\{(y, f_0(x_\alpha) - \epsilon_\alpha)\} \subset \Pi$ and $(y, r) = \lim (y, f_0(x_\alpha) - \epsilon_\alpha) \in \bar{\Pi}$. The inclusion $\Pi \subset \text{epi } V$ holds from the definition of the sets Π and epi V.

Now, let $z \in \mathbf{R}^l$. Then, the perturbed problem is defined by

$$(P_z) := \inf f_0(x),$$

$$\text{subject to } x \in C, \qquad f_i(x) \leq z_i, \; i \in I.$$

The next theorem provides a characterization result for strong regularity and conditions for existence of primal optimal solutions.

Theorem 3.1. Characterization of Strong Regularity. The program (P) is strongly regular if and only if the value function $V(\cdot)$ is lower semicontinuous on $\mathbf{R}^I$ and (P_z) attains its minimum, whenever $V(z)$ is finite.

Proof. ($\Rightarrow$) Assume that (P) is strongly regular. Then, $V(\cdot)$ is lower semicontinuous on $\mathbf{R}^I$, since epi $V=\Pi$ is closed. Let $V(z)$ be finite. Then, for each positive integer n, there exists $x_n \in C$, $f_i(x_n) \leq z_i$, $i \in I$, with $V(z)+1/n > f_0(x_n)$. Thus $(z, V(z)+1/n) \in \Pi$ and

$$(z, V(z)) = \lim_{n\to\infty} (z, V(z)+1/n) \in \bar{\Pi} = \Pi.$$

Hence, there exists $x_0 \in C$ such that $f_0(x_0) = V(y)$ and $f_i(x_0) \leq z_i$, $i \in I$.

($\Leftarrow$) Assume that $V(\cdot)$ is lower semicontinuous and that (P_z) attain its minimum, when $V(z)$ is finite. Let $(z, r) \in \bar{\Pi}$. Since $\Pi \subset \text{epi } V \subset \bar{\Pi}$, $\bar{\Pi} = \overline{\text{epi}}\ V = \text{epi } V$, and so $V(z) \leq r$. If $-\infty < V(z) < +\infty$, then by the assumption, there exists $x_0 \in C$, $f_i(x_0) \leq z_i$, $i \in I$ such that $V(z) = f_0(x_0) \leq r$. Hence, $(z, r) \in \Pi$. If $V(z) = -\infty$, then by the construction of $V(z)$, there exists $x \in C$, $f_i(x) \leq z_i$, $i \in I$ such that $f_0(x) \leq r$, which implies $(z, r) \in \Pi$. Hence, Π is closed, and so (P) is strongly regular. □

Proposition 3.4. Let the program (P) be convex-like. If the value function V is lower semicontinuous on $\mathbf{R}^I$, then it is convex.

Proof. Since (P) is convex-like, the set Π is midpoint convex. Then, by Lemma 2.1, the set $\bar{\Pi} = \overline{\text{epi } V} = \text{epi } V$ is convex, since $\Pi \subset \text{epi } V \subset \bar{\Pi}$ and V is lower semicontinuous on $\mathbf{R}^I$. Hence, V is convex. □

We finish this section by noting that the value function of a strongly regular convex-like problem is convex and that a strongly regular convex-like problem is F-convex-like.

4. Characterization of Zero Duality Gaps and ϵ-Subdifferentials

Let $\epsilon \geq 0$. The ϵ-subdifferential of a proper convex function $p: X \to \mathbf{R} \cup \{+\infty\}$ [i.e., p is not identically equal to $+\infty$ and $p(x) > -\infty$, for all $x \in X$] at $a \in X$, where $p(a) < +\infty$, is denoted by $\partial_\epsilon p(a)$ and is defined by

$$\partial_\epsilon p(a) = \{v \in X^*: p(x) \geq p(a) + \langle v, x-a\rangle - \epsilon, \forall x \in X\}.$$

Then the set $\partial_\epsilon p(a)$ is a weak* closed convex set in X^* which reduces to the subdifferential of p at a, for $\epsilon = 0$.

If p is a proper convex function, then p is lower semi-continuous at a if and only if $\partial_\epsilon p(a) \neq \emptyset$, whenever $\epsilon > 0$. If p is also positively homogeneous, then $\partial_\epsilon p(0) = \partial p(0)$, for all $\epsilon \geq 0$. For details see Hiriart-Urruty (Ref. 22). For the program (P), if $V(0)$ is finite and if $\partial_\epsilon V(0) \neq \emptyset$, then the value function satisfies the following relation: there exists $\lambda \in E$, such that

$$(S_\epsilon) \quad V(z) \geq V(0) - \langle \lambda, z \rangle - \epsilon, \qquad z \in \mathbf{R}^l,$$

and the vector λ can be interpreted as an approximate lower bound on the marginal rate of decrease in the optimal value of the objective function when the problem (P) is perturbed. Hence, we say that problem (P) is nearly stable if the value function V satisfies the relation (S_ϵ), for $\epsilon > 0$, and stable (Ref. 4) if (S_ϵ) holds for $\epsilon = 0$. In these cases, the components of λ are called approximate equilibrium prices and equilibrium prices, respectively (Refs. 23 and 24). We shall see that $\lambda \in \partial_\epsilon V(0)$ implies that $\lambda \in \Lambda$ as well.

We note that the regular F-convex-like (or the strongly-regular convex-like) program (P) is nearly stable, since $V(\cdot)$ is proper convex and lower semicontinuous at 0. If, in addition, the functions involved in (P) are positively homogeneous, then (P) is always stable, since $V(\cdot)$ is a proper sublinear functional. Moreover, if the regular F-convex-like program (P) satisfies the following calmness condition [see Clark, Ref. 21] that

$$\liminf_{\substack{y \to 0 \\ t \downarrow 0}} [V(ty) - V(0)]/t > -\infty \tag{9}$$

(this is true in particular if V is Lipschitz near 0 or V is bounded above on a neighborhood of 0), then by a theorem of Rockafeller (Ref. 25) [see also Borwein and Strojwas (Ref. 26)] and by our Propositions 3.1 and 3.3, (P) is stable.

The following proposition characterizes a zero duality gap for (P) and (D), extending a result of Geoffrion (Theorem 7, Ref. 27) to nonconvex infinite problems. Note that we do not require that V is proper convex and so cannot assume that a nonempty ϵ-subdifferential implies that V is lower semicontinuous. We define clco V as the function with epigraph given by the closure of the convex hull of the epigraph of V; see Ref. 5, where it is shown that (ii) and (iii) below are equivalent.

Theorem 4.1. Consider problems (P) and (D). Assume that $V(0)$ is finite. Then, the following statements are equivalent:

(i) (P) is nearly stable;
(ii) there is zero duality gap between (P) and (D);
(iii) clco $V(0) = V(0)$.

Proof. (i)⇒(ii). Let $\epsilon > 0$. From (i), there is $\lambda \in Y^*$ such that, for all $z \in \mathbf{R}^l$,

$$V(0) \leq V(z) + \langle \lambda, z \rangle + \epsilon.$$

If $x \in C$, then by choosing $\bar{z}_i = f_i(x)$, for $i \in I$, we have $f_i(x) \leq \bar{z}_i$ and, for $\lambda \in E$,

$$f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \geq V(\bar{z}) + \langle \lambda, \bar{z} \rangle \geq V(0) - \epsilon.$$

Hence,

$$\inf_{x \in C} f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \geq V(0) - \epsilon.$$

If $\lambda \in (\mathbf{R}^l_+)^* := \Lambda$, then

$$\sup_{\lambda \in \Lambda} \inf_{x \in C} f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \geq V(0),$$

since $\epsilon > 0$ is arbitrary. The reverse inequality follows from weak duality.

It remains to show that $\lambda \in \Lambda$. Suppose the contrary. Then, by a separation theorem, there exists $u \in \mathbf{R}^l_+$ with $\langle \lambda, u \rangle = -1$, since $\mathbf{R}^{l*}_+$ is a closed convex cone. So,

$$\begin{aligned} V(2\epsilon u) &\geq V(0) - \langle \lambda, 2\epsilon u \rangle - \epsilon \\ &= V(0) + 2\epsilon - \epsilon \\ &= V(0) + \epsilon. \end{aligned}$$

But since $2\epsilon u \in \mathbf{R}^l_+$, we have that $V(2\epsilon u) \leq V(0)$, a contradiction.

(ii)⇒(i). Assume that there is no duality gap between (P) and (D). Let $\epsilon > 0$. Then, there exists $\lambda \in \Lambda$ such that

$$\inf_{x \in C} f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \geq V(0) - \epsilon.$$

Thus, for all $x \in C$,

$$f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \geq V(0) - \epsilon.$$

If $f_i(x) \leq z_i$, $i \in I$, then

$$-\sum_{i \in I} \lambda_i f_i(x) + \sum_{i \in I} \lambda_i z_i \geq 0,$$

and so

$$f_0(x) \geq V(0) - \sum_{i \in I} \lambda_i z_i - \epsilon.$$

Hence,

$$V(z) \geq V(0) - \sum_{i \in I} \lambda_i z_i - \epsilon,$$

and (P) is nearly stable.

That (ii) and (iii) are equivalent is proved in Ref. 5 using the properties of conjugate functions. Note that, since (i) and (iii) are clearly equivalent, the above provides a proof of the equivalence of (ii) and (iii) without conjugate functions. □

Remark 4.1. We observe from the above theorem that the zero duality gap problem can be viewed geometrically as the (strong) separation property of the closure of the epigraph of V and the point $(0, V(0)-\epsilon)$, for every $\epsilon>0$. This is in contrast with strong duality, which is obtained with a supporting hyperplane with $\epsilon=0$. For various characterizations and detailed classification schemes for linear and convex programming duality results, we refer the reader to Refs. 28-30. It is known that, even if (P) attains its minimum and there is zero duality gap between (P) and (D), problem (D) does not, in general, attain its maximum, unless some additional regularity conditions are satisfied. The following example illustrates this situation where a primal convex infinite program attains its minimum, but the dual program does not attain its maximum. However, there is zero duality gap between the primal and the dual programs.

Example 4.1. Consider the problem

$$\inf_{(x,y)\in R^2} x^2+y,$$

subject to $x\leq 0$, $-y-1\leq 0$,

$$x/i-y\leq 0, \qquad i=3,4,\ldots.$$

The infimum for the problem is attained at $(x,y)=(0,0)$. The Lagrangian dual for the problem is the problem

$$\sup\left\{\psi(\lambda):=\inf_{x,y} x^2+y+\lambda_1 x-\lambda_2(y+1)+\sum_{i=3}^{\infty}\lambda_i(x/i-y)\,\middle|\,\lambda_i\geq 0\right\},$$

with only finitely many nonzero λ_i. Choosing

$$\lambda_i=1, \qquad \text{for } i=n\geq 3,$$

$$\lambda_i=0, \qquad \text{for } i\neq n,$$

$\psi(\lambda)$ becomes

$$\psi(\lambda)=-1/(4n^2),$$

so

$$\psi(\lambda)\to 0, \qquad \text{as } n\to\infty.$$

Hence, value(dual) $=0$, and there is zero duality gap. However, the dual problem does not attain its maximum.

If the regular F-convex-like program (P) satisfies the additional calmness condition (9), then (P) is stable and so there is zero duality gap between (P) and (D), and (D) attains its maximum. In this case, $\partial V(0) \neq \varnothing$ and

$$\partial V(0) = \left\{ -\lambda : \lambda \in \Lambda \text{ and } V(0) = \inf_{x \in C} f_0(x) + \sum_{i \in I} \lambda_i f_i(x) \right\}.$$

The latter equation holds for the regular F-convex-like program (P) in which the functions f_i, $i \in \bar{I}$ are positively homogeneous, without the additional calmness hypothesis, since V is sublinear.

Let $\epsilon > 0$. We recall that the feasible λ of (D) is called the ϵ-optimal solution to (D) if $\phi(\lambda) > \nu - \epsilon$. As a consequence of Theorem 4.1, we obtain a characterization of the ϵ-subdifferentials of the value function for a regular convex-like program in terms of ϵ-optimal solutions of the dual program.

Corollary 4.1. If (P) is a regular convex-like program and $\epsilon > 0$, then $\partial_\epsilon V(0)$ is nonempty and

$$\partial_\epsilon V(0) = \{-\lambda \mid \lambda \text{ is an } \epsilon\text{-optimal solution to (D)}\}.$$

Proof. This is immediate from the proof of Theorem 4.1. □

We now use Theorem 4.1 to show a relationship between ϵ-subgradients of the value function and Lagrange multiplier vectors for an ϵ-optimal solution of an infinitely constrained convex program.

Corollary 4.2. Assume in (P) that $C = X$, the functions f_i, $i \in I \cup \{0\}$ are convex, and one of them is continuous on X. If (P) is regular, then for every $\epsilon > 0$,

$$\begin{aligned} \partial_\epsilon V(0) \subset \Big\{ -\lambda \mid \lambda \in \Lambda,\ \exists \bar{x} \in X, &\text{ a finite number of points } t_1, \ldots, t_n \text{ in } I, \\ &\epsilon_i \geq 0,\ i = 0, 1, 2, \ldots, n,\ \epsilon = \sum_{i=0}^{n} \epsilon_i, \\ &0 \in \partial_{\epsilon_0} f_0(\bar{x}) + \sum_{i=1}^{n} \partial_{\epsilon_i}(\lambda_i f_{t_i})(\bar{x}),\ -\epsilon \leq \sum_{i=1}^{n} \lambda_i f_{t_i}(\bar{x}) \leq 0 \Big\}. \end{aligned} \tag{10}$$

Proof. Since (P) is regular, (P) attains a minimum at some feasible point $\bar{x} \in X$; thus, $V(0) = f_0(\bar{x})$. Let $\epsilon > 0$ and $-\lambda \in \partial_\epsilon V(0)$. Then, as in the proof of Theorem 4.1,

$$\inf_{x \in X} f_0(x) + \sum_{i=1}^{n} \lambda_i f_{t_i}(x) \geq f_0(\bar{x}) - \epsilon,$$

for some $t_1, \ldots, t_n \in I$, since $\lambda_j \neq 0$, for only finitely many $j \in I$. Since $\bar{x}$ is feasible,

$$\inf_{x \in X} f_0(x) + \sum \lambda_i f_{t_i}(x) \geq f_0(\bar{x}) + \sum_{i=1}^{n} \lambda_i f_{t_i}(\bar{x}) - \epsilon.$$

By the definition of the ϵ-subdifferential,

$$0 \in \partial_\epsilon \left(f_0(\bar{x}) + \sum_{i=1}^{n} \lambda_i f_{t_i}(\bar{x}) \right).$$

Now, by Proposition 1.3 of Hiriart-Urruty (Ref. 24),

$$\partial_\epsilon \left(f_0(\bar{x}) + \sum_{i=1}^{n} \lambda_i f_{t_i}(\bar{x}) \right) = \bigcup_{\substack{\epsilon_i \geq 0 \\ \sum_{i=0}^{n} \epsilon_i = \epsilon}} \partial_{\epsilon_0} f_0(\bar{x}) + \sum_{i=1}^{n} \partial_{\epsilon_i} (\lambda_i f_{t_i})(\bar{x}).$$

Hence, λ belongs to the right-hand side of the inclusion (10). □

5. Convex and Generalized Linear Programs

In this section, we obtain zero duality gap results for semi-infinite convex and generalized linear programming problems as a specialization of our main theorem of Section 2. We begin by fixing some preliminaries about recession cones. Let A be a nonempty subset of X. Following Rockafellar (Ref. 31) and Dedieu (Ref. 32), the recession cone of A, denoted by rec A, is defined by

$$\operatorname{rec} A = \{x \in X : \exists \{x_\alpha\} \subset A \text{ and } \{\lambda_\alpha\} \subset \mathbf{R}_+, \lambda_\alpha \to 0, \lambda_\alpha x_\alpha \to x\}.$$

It is clear from the definition that rec A is a closed cone containing 0.

We see in the following proposition that the boundedness of a set in a finite-dimensional space is characterized in terms of its recession cone.

Proposition 5.1. Let $X = \mathbf{R}^n$ and $A \subset \mathbf{R}^n$. Then, A is bounded if and only if $\operatorname{rec} A = \{0\}$.

Proof. If A is bounded, then clearly $\operatorname{rec} A = \{0\}$. Conversely, assume that $\operatorname{rec} A = \{0\}$. Suppose that A is unbounded. Then, there exists a sequence $\{x_n\}$ in A such that $x_n \neq 0$, for all n, and $\|x_n\| \to \infty$. Let $u_n = x_n / \|x_n\|$. Then, $\{u_n\}$ is a sequence of points in the unit ball

$$S = \{x \in \mathbf{R}^n : \|x\| = 1\}.$$

Since S is compact, there exists a subsequence $\{u_n\}$ which converges to $u \in S$. So, $u \neq 0$. This is a contradiction, since

$$u = \lim_n u_n = \lim_n [1/\|x_n\|]x_n \in \operatorname{rec} A = \{0\}. \qquad \square$$

Proposition 5.2. (i) If A is a convex set, then

$$0^+A = \{x \in X: a + tx \in A, \text{ for all } t \geq 0 \text{ and all } a \in A\} \subset \operatorname{rec} A.$$

(ii) If A is closed and convex, then $\operatorname{rec} A = 0^+A$, and $\operatorname{rec} A$ is a closed convex cone.

Proof. (i) Let $x \in 0^+A$. Then, for fixed $a \in A$, $a + nx = x_n \in A$ and so

$$x = \lim_n (1/n)(a + nx) = \lim_n (1/n)x_n \in \operatorname{rec} A.$$

(ii) Assume that A is closed and convex. Then, it suffices to prove that $\operatorname{rec} A \subset 0^+A$. Let $x \in \operatorname{rec} A$. Then, there exist nets $\{x_\alpha\} \subset A$ and $\{\lambda_\alpha\} \subset \mathbf{R}_+$, $\lambda_\alpha \to 0$, such that $\lambda_\alpha x_\alpha \to x$. For fixed $t \geq 0$, choose a subnet $\{\lambda_\gamma\}$ such that $t\lambda_\gamma \leq 1$. Since A is convex, for each $t \geq 0$ and each $a \in A$,

$$y_\gamma = (1 - t\lambda_\gamma)a + \lambda_\gamma t x_\gamma \in A.$$

Since A is closed and $y_\gamma \to a + tx$, $a + tx \in A$, and so $x \in 0^+A$. The convexity of $\operatorname{rec} A$ follows from the convexity of 0^+A. $\square$

In the following theorem, we present the Clark-Duffin theorem due to Karney (Theorem 2, Ref. 20) under more general conditions.

Theorem 5.1. For problem (P), assume that $X = \mathbf{R}^n$, C is a closed convex subset of X, and the functions f_i, for all $i \in \tilde{I}$, are convex and lower semicontinuous. If (P) is consistent and has the finite value μ and if there exist $\gamma > \mu$ and $\epsilon > 0$ such that

$$\operatorname{rec}\{x \in C: f_0(x) \leq \gamma\} \cap \operatorname{rec}\{x \in C: f_i(x) \leq \epsilon, i \in I\} = \{0\},$$

then there is a zero duality gap for (P) and (D).

Proof. The conclusion will follow from Theorem 2.1 and Propositions 2.2 and 5.1 if we show that

$$\operatorname{rec}\{x \in C: f_i(x) \leq \epsilon, i \in I\} = \{0\}$$

is equivalent to

$$\operatorname{rec}\{x \in C: f_i(x) \leq \epsilon, i \in I_0\} = \{0\},$$

for some finite set I_0 of I. It suffices to show that

$$\operatorname{rec}\{x \in C: f_i(x) \leq \epsilon, i \in I_0\} = \{0\}.$$

Suppose that

$$\operatorname{rec}\{x \in C: f_i(x) \le \epsilon, \forall i \in I_\alpha\} \ne \{0\},$$

for every finite subset I_α of I. Then, there exists a sequence $\{y_\alpha\}$ of nonzero elements of $\mathbf{R}^n$ such that

$$y_\alpha \in \operatorname{rec}\{x \in C: f_i(x) \le \epsilon, \forall i \in I_\alpha\}.$$

Without loss of generality, we can assume that $\|y_\alpha\| = 1$. Choose a subsequence of y_α which converges to y_0. Now, for any $i \in I$, for all $\lambda > 0$, and for $x \in \{y \in C: f_i(y) \le \epsilon\}$,

$$f_i(x + \lambda y_\alpha) \le \epsilon,$$

for sufficiently large α. Hence,

$$f_i(x + \lambda y_0) \le \epsilon;$$

thus,

$$y_0 \in \operatorname{rec}\{x \in C: f_i(x) \le \epsilon, \forall i \in I\},$$

a contradiction. □

Remark 5.1. If the feasible set of the convex semi-infinite program (P) is bounded, then

$$\{0\} = \operatorname{rec}\{x \in C: f_i(x) \le 0, i \in I\} = \operatorname{rec}\{x \in C: f_i(x) \le \epsilon, i \in I\},$$

and hence our recession condition in Theorem 5.1 holds.

We now apply Theorem 2.1 to obtain a zero duality gap theorem for the following generalized linear program:

(SP) $\inf f_0(x)$,

subject to $x \in C$, $\quad g_i(x) \le b_i$, $\quad i \in I$;

here, $C \subset X$ is a closed convex cone; f_0 and, for each $i \in I$, $g_i: C \to \mathbf{R}$ are convex, positively homogeneous and lower semicontinuous functions; and $b_i \in \mathbf{R}$, for $i \in I$. The dual program for (SP) is the following program:

(SD) $\sup_{\lambda \in \Lambda} \inf_{x \in C} f_0(x) + \sum_{i \in I} \lambda_i (g_i(x) - b_i)$.

Following Anderson and Nash (Ref. 33), a closed cone C is said to have a compact sole if there exists a compact set B such that

$$0 \notin B \quad \text{and} \quad C = \bigcup_{\lambda \ge 0} \lambda B.$$

Theorem 5.2. Assume that problem (SP) has a finite value and is consistent. If C has a compact sole and there is no $x \in C$, other than zero with $f_0(x) \le 0$ and $g_i(x) \le 0$, $i \in I$, then there is a zero duality gap for (SP) and (SD).

Proof. The conclusion will follow from Theorem 2.1 by taking $f_i(x) = g_i(x) - b_i$, for $i \in I$, if we show that problem (SP) is strongly regular. We shall prove that the set

$$\tilde{\Pi} := \{(y, r) \in \mathbf{R}^I \times \mathbf{R}: \exists x \in C, f_0(x) \leq r, g_i(x) \leq b_i + y_i, i \in I\}$$

is closed. Let $\{(y^t, r^t)\}$ be a net in $\tilde{\Pi}$ which converges to $(y, r) \in \mathbf{R}^I \times \mathbf{R}$. Since C has a compact sole, there exists a compact set B, $0 \notin B$ such that

$$C = \bigcup_{\lambda \geq 0} \lambda B.$$

Then, for each t, $x^t \in C$, $\lambda^t \geq 0$, and $z^t \in B$ such that $x^t = \lambda^t z^t$,

$$f_0(x^t) \leq r^t \quad \text{and} \quad g_i(x^t) \leq b_i + y_i^t, \qquad i \in I. \tag{11}$$

We shall first show that the net $\{\lambda^t\}$ is bounded. Suppose that $\{\lambda^t\}$ is unbounded. Then, without loss of generality, we can assume that $\lambda^\beta \to \infty$ for some subnet $\{\lambda^\beta\}$ and that all λ^β are positive and $\{z^\beta\} \subset B$. Let $z^\beta \to z \in B$. Then, $z \neq 0$ and $z \in C$. From (11) and the positive homogeneity hypothesis,

$$f_0(z^\beta) \leq (\lambda^\beta)^{-1} r^\beta,$$
$$g_i(z^\beta) \leq (\lambda^\beta)^{-1}[b_i + y_i^\beta], \qquad i \in I.$$

Then, by lower semicontinuity,

$$f_0(z) \leq 0 \quad \text{and} \quad g_i(z) \leq 0, \qquad i \in I.$$

This is a contradiction to our assumption.

Now, we can choose subnets $\{\lambda^\alpha\}$ and $\{z^\alpha\}$ such that $\lambda^\alpha \to \hat{\lambda}$ and $z^\alpha \to \hat{z}$. Let

$$\lim_\alpha x^\alpha = \hat{\lambda}\hat{z} = \hat{x} \in C.$$

From (11) and lower semicontinuity,

$$f_0(\hat{x}) \leq r \quad \text{and} \quad f_i(\hat{x}) \leq y_i, \qquad i \in I,$$

and hence $(y, r) \in \tilde{\Pi}$. □

6. Conclusions

As published papers in the mathematical programming literature confirm, zero duality gap results are significant and have played an important role in infinite-dimensional linear and convex programming. In this paper, we have shown that zero duality gap results are not limited to convex or linear infinite programs, but continue to hold for a much larger class of nonconvex (convex-like) infinite programs. In particular, the class of convex-like programs includes certain composite nonconvex programs. We established relationships between our new regularity condition and generalized continuity and differentiability properties of the value function for convex-like programs.

We also note that several characterizations and classification schemes for the duality of linear and convex programming problems are known (e.g., see Refs. 28-30). Here, we added to these schemes one more simple characterization for zero duality gap using the ϵ-subdifferential of the value function without convexity.

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