Invariant Ellipsoidal Cones

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ABSTRACT

Conditions on the spectrum of a matrix A which are equivalent to the existence of a proper convex cone K such that A is K-nonnegative, K-positive, K-irreducible, or K-strongly nonnegative are known. We study the effect of adding the requirement that K be ellipsoidal. It is shown that the conditions are unchanged, except in the K-nonnegative case.

1. INTRODUCTION

J. Vandergraft [8] and L. Elsner [2] independently extended the classical Perron-Frobenius theory of nonnegative matrices by deriving spectral conditions on a given $n \times n$ matrix A which are necessary and sufficient for the

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existence of a proper cone $K \subset \mathbb{R}^n$ such that A is K-nonnegative. By a proper cone K we mean one which is closed, convex, pointed $(\mathbf{K} \cap \{-\mathbf{K}\} = \{0\})$, and such that the interior $\operatorname{int} K \neq \emptyset$. They also gave characterizations in which the stronger properties of K-positivity and K-irreducibility replace K-nonnegativity. Furthermore, Elsner [2] gave conditions on A which are equivalent to the existence of a proper cone $\mathbf{K}$ such that A is strongly K-nonnegative; that is, A is K-nonnegative and has an eigenvector in the interior of K.

In this paper, we investigate how the aforementioned characterizations are affected if we add the requirement that the proper cone K be ellipsoidal; i.e., K has ellipsoidal cross sections. Unlike the polyhedral cones, the ellipsoidal cones have a smooth boundary, but, like the polyhedral cones, the ellipsoidal cones can be handled algebraically using the solution set of an inequality. The ellipsoidal cones form a subset of the set of rotund cones, i.e. the cones whose only proper faces are the one-dimensional extreme rays. We show that the characterizations are unchanged in the stronger than K-nonnegativity cases, but change in the ordinary K-nonnegativity case. For example, if the degree (the size of the largest Jordan block) of the spectral radius of A is greater than 3, then there cannot exist an ellipsoidal cone K such that A is K-nonnegative.

Section 2 presents several preliminaries. Then in Section 3 we deal with the K-positive, K-irreducible, and strongly K-nonnegative cases. It will be shown that in these situations, the added imposition that K be ellipsoidal does not alter the characterizations derived by Vandergraft and Elsner, in spite of the fact that the constructive proofs given by those authors can yield nonellipsoidal (and in fact polyhedral) cones. In Sections 4 and 5 we consider the more difficult K-nonnegative case. Our conditions characterizing the existence of an ellipsoidal proper cone such that A is K-nonnegative are more restrictive than those of Vandergraft and Elsner, where K was not assume to be ellipsoidal. Many of our results hold for the more general class of rotund cones. However, the characterization in the K-nonnegative case, given in Theorem 5.1, is specific to the class of ellipsoidal cones.

2. PRELIMINARIES

We first present some preliminary definitions and results.

A set K in $\mathbb{R}^n$ is a *cone* if $\lambda K \subset K$ for all $\lambda \geqslant 0$. The cone K is called *proper* if it is closed, is convex, has nonempty interior, and is *pointed*, i.e. $\mathbf{K} \cap \{-\mathbf{K}\} = \{0\}$. The cone K is *polyhedral* if it is the intersection of a finite number of half spaces; i.e., there exist $\varphi^1,\dots,\varphi^k \in \mathbb{R}^n$ such that

$$K=\{x \in \mathbb{R}^n : (\varphi^i)^t x \leqslant 0,\ i=1,\dots,k\}. \tag{2.1}$$

The convex cone $L \subset \mathbf{K}$ is a *face* of K if

$$x,y \in K, \quad x+y \in L \quad \Rightarrow \quad x,y \in L.$$

The proper cone K is *rotund* if every face of $\mathbf{K}$, other than $\mathbf{K}$ itself and $\{0\}$, is a half line (called an *extreme ray*).

The (proper) cone K is *ellipsoidal* if there exists a symmetric, nonsingular, $n \times n$ real matrix Q, with exactly one negative eigenvalue $\lambda_n < 0$, and corresponding unit eigenvector u^n, such that

$$K=\{x \in \mathbb{R}^n : x^t Q x \leqslant 0,\ (u^n)^t x \geqslant 0\}. \tag{2.2}$$

In the sequel, an ellipsoidal cone K always has an associated matrix $Q=Q^t$ with eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_{n-1} > 0 > \lambda_n$ and corresponding eigenvectors $u^1,\dots,u^n$. Thus Q has inertia $(n-1,0,1)$, where the *inertia* is defined as the triple (p,z,n) denoting the number of positive, zero, and negative eigenvalues, respectively. Moreover, we let $\Lambda = U^t Q U = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$ denote the orthogonal diagonalization of Q. Note that

$$K \cup (-K) = \{x \in \mathbb{R}^n : x^t Q x \leqslant 0\}, \tag{2.3}$$

and that the orthogonal complement $\{u^n\}^{\perp}$ is a supporting hyperplane to both K and $-K$.

If K is an ellipsoidal cone with corresponding Q, λ_i, and u^i, $i=1,\dots,n$, then for $\alpha > 0$,

$$S_\alpha = \{x \in K : (u^n)^t x = \alpha\}$$

will be called an *ellipsoid* in the hyperplane

$$Z_\alpha = \{x : (u^n)^t x = \alpha\}.$$

Note that upon expanding Q in terms of its spectral decomposition (see e.g.

[3]) and x as $x = \sum_{i=1}^{n} \alpha_i u^i$, we obtain $\alpha = \alpha_n$ and

$$S_\alpha = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n-1} \lambda_i (\alpha_i)^2 \leqslant -\alpha^2 \lambda_n \right\}.$$

The eigenvector u^n lies in the interior of the ellipsoidal cone K. The hyperplanes Z_α are translations of $\{u^n\}^\perp$, and they intersect the cone to form ellipsoids. Moreover, each nonzero point $z \in \partial K$, the boundary of K, has a unique supporting hyperplane given by $H = \{Qz\}^\perp$, and $H \cap K$ is the extreme ray through z. (See Lemma 4.1 below.) Thus K is a rotund cone.

We will now show that an ellipsoidal cone can be defined using supporting hyperplanes other than $\{u^n\}^\perp$.

PROPOSITION 2.1. *Suppose that K is as above. Then*

$$K = \{ x \in \mathbb{R}^n : x^t Q x \leqslant 0,\ v^t x \geqslant 0 \} \tag{2.4}$$

if v satisfies

$$\{v\}^\perp \cap \{ x \in \mathbb{R}^n : x^t Q x \leqslant 0 \} = \{0\} \quad \textit{and} \quad v^t u^n \geqslant 0. \tag{2.5}$$

In particular, $v = u^n$ *satisfies* (2.5).

Proof. Let $x \in \mathbb{R}^n$ and $x = \sum_{i=1}^{n} \alpha_i u^i$. If $v = u^n$ in (2.5), we see that $x \in \{v\}^\perp$ implies that $\alpha_n = 0$. Moreover, since $\lambda_i > 0$, $i \neq n$, we have that $x^t Q x \leqslant 0$ implies $x = 0$. Thus $v = u^n$ satisfies (2.5). In view of (2.3), if v satisfies (2.5), then $\{v\}^\perp$ is a hyperplane separating K and $-K$. Therefore (2.4) holds. ■

A particular ellipsoidal cone which proves to be extremely useful is the so-called *ice-cream cone*

$$K_n = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{n-1} (x_i)^2 \right)^{1/2} \leqslant x_n \right\}. \tag{2.6}$$

PROPOSITION 2.2. *Let*

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix},$$

where I_k denotes the $k \times k$ identity matrix. Then K_n is the ellipsoidal cone with matrix Q and eigenvector $u^n = e^n$, the nth unit vector.

Proof. Clearly $x \in K_n$ if and only if $x^tQx \leqslant 0$ and $(e^n)^t x \geqslant 0$. ■

We will now show that every ellipsoidal cone is a nonsingular linear transformation of the ice-cream cone.

PROPOSITION 2.3. *K is an ellipsoidal cone if and only if $K = TK_n$ for some nonsingular matrix T. In particular, for an ellipsoidal cone K, we have $T = UD$, where D is the diagonal matrix with diagonal elements*

$$d_i = (1/|\lambda_i|)^{1/2}, \qquad i = 1, \dots, n,$$

and $\Lambda = U^tQU$ is the orthogonal diagonalization of Q.

Proof. Let K be an ellipsoidal cone and $T = UD$ be defined as above. Let

$$J = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.7}$$

Then

$$Q = U\Lambda U^t = UD^{-1}JD^{-1}U^t = (T^{-1})^t JT^{-1}. \tag{2.8}$$

Now, $x \in K$ if and only if $x^tQx \leqslant 0$, $(u^n)^t x \geqslant 0$, or equivalently

$$x = Ty, \qquad y^tJy \leqslant 0, \qquad (u^n)^t Ty \geqslant 0.$$

This in turn is equivalent to

$$x = Ty, \qquad y \in K_n, \tag{2.9}$$

since $(u^n)^tT = (u^n)^tUD = (1/|\lambda_n|)^{1/2}(e^n)^t$, which shows that $K = TK_n$.

Conversely, suppose that T is given. We need to show that TK_n is an ellipsoidal cone. Now,

$$TK_n = \{y : y^t(T^{-1})^t JT^{-1}y \leqslant 0, (e^n)^t T^{-1}y \geqslant 0\}.$$

By Sylvester's theorem (see e.g. [3]), the matrix $(T^{-1})^t J T^{-1}$ has the same inertia as J. To see that TK_n is an ellipsoidal cone, note that $v = (T^{-1})^t e^n$ satisfies (2.5) for $Q = (T^{-1})^t J T^{-1}$ and one of the two possible choices for u^n. ■

An immediate consequence of Propositions 2.2 and 2.3 is the following.

COROLLARY 2.1. *If $K \subset \mathbb{R}^n$ is an ellipsoidal cone and T is a nonsingular $n \times n$ matrix, then TK is an ellipsoidal cone.*

3. THE POSITIVE, IRREDUCIBLE, AND STRONGLY NONNEGATIVE CASES

In the present work it will be convenient to employ the *real canonical form* $C(A)$ of a real $n \times n$ matrix A. Recall that $C(A)$ is a block-diagonal matrix which is unique up to the order of the blocks and is a *real* similarity transformation of A. For a real eigenvalue λ, these blocks can either be order-1 blocks

$$B(\lambda;1) = [\lambda], \tag{3.1}$$

or order-k blocks of the form

$$B(\lambda;k) = \begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda \end{bmatrix}.$$

Not let $\lambda = a + ib$ be a complex eigenvalue of A with $b > 0$. Then associated with λ (and with the complex conjugate $\bar{\lambda}$ as well), $C(A)$ contains either order-2 blocks of the form

$$B(a,b;2) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

or order-$2k$ blocks

$$B(a,b;2k)=\begin{bmatrix} B(a,b;2) & & & \\ I_2 & \ddots & & \\ & \ddots & \ddots & \\ & & I_2 & B(a,b;2) \end{bmatrix}. \tag{3.2}$$

Let $\sigma(A)$ denote the spectrum and $\rho(A)$ the spectral radius of A. In this paper we adopt the following conventions: If $\lambda\in\sigma(A)$, then all the blocks associated with λ in $C(A)$ are (diagonally) adjacent with larger blocks below smaller blocks. Furthermore, if $\lambda\in\sigma(A)$, $\mu\in\sigma(A)$, and $|\lambda|>|\mu|$, then the blocks associated with λ are below those for μ.

We can now conclude the following.

Corollary 3.1. *The matrix A leaves an ellipsoidal cone invariant if and only if any matrix similar to A, and C(A) in particular, leaves an ellipsoidal cone invariant.*

Proof. This follows from Corollary 2.1, since $AK\subset K$ and $A=S^{-1}BS$ if and only if $B(SK)\subset(SK)$. ■

We shall denote by $d(\lambda)$ the *degree* of an eigenvalue λ; this is the order of its largest block in the Jordan canonical form of A. Note that if λ is real, then $d(\lambda)$ is also the order of the largest block for λ in $C(A)$, and if λ is complex, then $2d(\lambda)$ is the order of the largest block in $C(A)$ associated with λ (and with $\bar{\lambda}$).

The Euclidean norm of a vector $x\in\mathbb{R}^n$ is denoted by $\|x\|$. For an $n\times n$ matrix B, we denote by $\|B\|$ the associated operator norm

$$\|B\|=\max\{\|Bx\|:\|x\|=1\}=\left[\rho(B^tB)\right]^{1/2}.$$

We shall require the following estimates for the norms of real canonical blocks:

Lemma 3.1.

(a) $\|B(a;1)\|^2=a^2$.
(b) $\|B(a;2)\|^2\leqslant a^2+|a|+1$.
(c) $\|B(a;k)\|^2\leqslant a^2+2|a|+1$ *for* $k\geqslant 3$.
(d) $\|B(a,b;2)\|^2=a^2+b^2$.
(e) $\|B(a,b;4)\|^2\leqslant a^2+b^2+|a|+|b|+1$.
(f) $\|B(a,b;2k)\|^2\leqslant a^2+b^2+2(|a|+|b|)+1$ *for* $k\geqslant 3$.

Proof. The equality in (a) is trivial, while (d) follows from the fact that $B(a,b;2)^t B(a,b;2)$ is a diagonal matrix with diagonal entries a^2+b^2. The inequalities in (b), (c), (e) and (f) all follow directly from a straightforward application of Gersgorin's theorem (see e.g. [3]) to the matrix B^tB, where B is the particular block under consideration. ■

A real $n\times n$ matrix A is elementwise positive if and only if $A(\mathbb{R}^n_+\setminus\{0\})\subset \operatorname{int}\mathbb{R}^n_+$, where $\mathbb{R}^n_+$ denotes the nonnegative orthant. The following definition generalizes this idea to proper cones other than $\mathbb{R}^n_+$.

DEFINITION 3.1. Let A be a real $n\times n$ matrix, and let $K\subset\mathbb{R}^n$ be a proper cone. Then A is *K-positive* (denoted $A \overset{K}{\gg} 0$) provided that $A(K\setminus\{0\})\subset \operatorname{int} K$. We let Π^P (Π^P_e) denote the set of $n\times n$ matrices A for which there exists a proper (an ellipsoidal) cone K such that $A\overset{K}{\gg} 0$.

The following result appeared in [8] and [2].

THEOREM 3.1. *Let A be a real $n\times n$ matrix with $n>1$. Then the following are equivalent:*

$$A\in\Pi^P; \tag{3.3}$$

The spectral radius of A, $\rho(A)$, is a simple eigenvalue of A, greater than the modulus of any other eigenvalue. (3.4)

In [8] and [2], the implication (3.4) ⇒ (3.3) in the above theorem is proven by construction of a proper cone K such that A is K-positive. In general, this cone is not ellipsoidal. (Actually, if the spectrum of A is real, then K is polyhedral.) The following result asserts that (3.3) and (3.4) are in fact equivalent to the existence of an ellipsoidal cone K such that $A\overset{K}{\gg} 0$. (This result was also proved in [1] using results on matrix norms.)

THEOREM 3.2. *For fixed n we have*

$$\Pi^P=\Pi^P_e. \tag{3.5}$$

Proof. Let A satisfy (3.4). In view of Theorem 3.1, our only goal is to show that $A\in\Pi^P_e$. Due to Corollary 3.1, we can assume that A is in real canonical form. We shall in fact prove that (3.4) implies $A\overset{K_n}{\gg} 0$.

Write the set of distinct eigenvalues of A as

$$\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}, \tag{3.6}$$

where

$$\lambda_1 = \rho(A) \quad \text{and} \quad |\lambda_j| \geq |\lambda_p| \quad \text{if } j < p. \tag{3.7}$$

Then

$$A = \begin{bmatrix} Q_m & & & \\ & \ddots & & \\ & & Q_2 & \\ & & & \rho(A) \end{bmatrix}, \tag{3.8}$$

where Q_j denotes the aggregate of all blocks for the eigenvalue λ_j.

Upon writing each eigenvalue as $\lambda_j = a_j + ib_j$ (where $b_j = 0$ for real eigenvalues), let us temporarily assume that

$$(a_j)^2 + (b_j)^2 + 2(|a_j| + |b_j|) + 1 < [\rho(A)]^2, \qquad j = 2, \ldots, m. \tag{3.9}$$

Let $0 \neq x \in K_n$. Then

$$Ax = \left((Q_m x^m)^t, \ldots, (Q_2 x^2)^t, \rho(A) x_n\right)^t, \tag{3.10}$$

where x has been partitioned as

$$x = \left(x^m, x^{m-1}, \ldots, x^2, x_n\right)$$

and where, for $j = 2, \ldots, m$, the number of components of x^j is equal to the order of Q_j. Then Lemma 3.1 implies that

$$\|Q_j\|^2 < [\rho(A)]^2, \qquad j = 2, \ldots, m. \tag{3.11}$$

Now since $x \in K_n$ and $x \neq 0$, it follows that $x_n \neq 0$ and

$$\sum_{j=2}^{m} \|Q_j x^j\|^2 \leqslant \sum_{j=2}^{m} \|Q_j\|^2 \|x^j\|^2 < [\rho(A)]^2 (x_n)^2, \tag{3.12}$$

which implies that $Ax \in \operatorname{int} K_n$.

Hence we have verified that $A \in \Pi_e^P$ in case (3.9) holds. Now suppose that (3.9) fails. Since, by our hypothesis, $\rho(A)$ strictly dominates every other eigenvalue in modulus, it follows that, for $\gamma > 0$ chosen sufficiently large, the matrix $C(\gamma A)$ will satisfy (3.9), since for each $j = 2, \dots, m$, we have

$$\lim_{\gamma \to \infty} \frac{[\gamma\rho(A)]^2}{(\gamma a_j)^2 + (\gamma b_j)^2 + 2(|\gamma a_j| + |\gamma b_j|) + 1} = \frac{[\rho(A)]^2}{(a_j)^2 + (b_j)^2} > 1. \tag{3.13}$$

In view of Corollary 3.1, $\gamma A \in \Pi_e^P$ if and only if $C(\gamma A) \in \Pi_e^P$, which in turn is equivalent to $A \in \Pi_e^P$. This completes the proof. ∎

Vandergraft [8] and Elsner [2] introduced the following generalizations of nonnegativity and irreducibility with respect to a proper cone.

Definition 3.2. Let $K \subset \mathbb{R}^n$ be a proper cone. Then a real $n \times n$ matrix is said to be *K-nonnegative* provided that $AK \subset K$. We denote this property by $A \overset{K}{\geqslant} 0$. We let Π^N (Π_e^N) denote the set of $n \times n$ matrices A for which there exists a proper (an ellipsoidal) cone K such that $A \overset{K}{\geqslant} 0$.

Definition 3.3. Let $K \subset \mathbb{R}^n$ be a proper cone. Then a real $n \times n$ matrix is said to be *K-irreducible* provided that $A \overset{K}{\geqslant} 0$ and A has no eigenvector in ∂K (the boundary of K). We let Π^I (Π_e^I) denote the set of $n \times n$ matrices A for which there exists a proper (an ellipsoidal) cone K such that A is K-irreducible.

The following result appeared in [8] and [2].

Theorem 3.3. *Let A be a real $n \times n$ matrix. Then the following are equivalent:*

$$A \in \Pi^I; \tag{3.14}$$

$\rho(A)$ is a simple eigenvalue of A and $d(\lambda) = 1$ for all $\lambda \in \sigma(A)$ such that $|\lambda| = \rho(A)$. (3.15)

We can now prove the following result.

THEOREM 3.4. *For fixed n we have*

$$\Pi^I = \Pi_e^I. \tag{3.16}$$

Proof. Because of Theorem 3.3, we need only verify that (3.15) implies $A \in \Pi_e^I$. By Corollary 3.1, we can assume that A is already in real canonical form. We again will use the notation of (3.8). Let us partition the index set

$$\{2,\dots,m\} = J \cup \hat{J},$$

where $|\lambda| < \rho(A)$ for $j \in J$ and $|\lambda| = \rho(A)$ for $j \in \hat{J}$. Since we are presently assuming that (3.15) holds, we have $d(\lambda_j) = 1$ if $j \in \hat{J}$, while there is no degree restriction on λ_i for $j \in J$. Let us temporarily assume that

$$(a_j)^2 + (b_j)^2 + 2(|a_j| + |b_j|) + 1 < [\rho(A)]^2 \qquad \text{for all} \quad j \in J. \tag{3.17}$$

Then Lemma 3.1 implies that

$$\|Q_j\|^2 < [\rho(A)]^2 \qquad \text{for all} \quad j \in J \tag{3.18}$$

and

$$\|Q_j\|^2 = [\rho(A)]^2 \qquad \text{for all} \quad j \in \hat{J}. \tag{3.19}$$

Now let $x \in K_n$. Similarly to (3.12), we have

$$\sum_{j=2}^{m} \|Q_j x^j\|^2 \leqslant [\rho(A)]^2 (x_n)^2, \tag{3.20}$$

which implies that $Ax \in K_n$. This shows that $A \overset{K_n}{\geqslant} 0$. Now observe that if a vector $v \in \mathbb{R}^n$ is an eigenvector of A corresponding to a real eigenvalue $\lambda \neq \rho(A)$, then v is necessarily a nonzero linear combination of a subset of the vectors $(e^i)_{i=1}^{n-1}$, where e^i denotes the ith column of the $n \times n$ identity matrix. Consequently, $v \notin K_n$. Since $\rho(A)$ is simple, the only real eigenvector of A in K_n (up to scalar multiples) is $e^n \in \operatorname{int} K_n$. Therefore A is K_n-irreducible, as required.

Now note that (3.17) holds for $C(\gamma A)$ when $\gamma > 0$ is sufficiently large. Similarly to the conclusion of the proof of Theorem 3.2, we apply Corollary 3.1 and conclude that $A \in \Pi_e^I$. ∎

We now introduce a concept which is more general than K-positivity or K-irreducibility.

DEFINITION 3.4. Let $K \subset \mathbb{R}^n$ be a proper cone. A real $n \times n$ matrix A is said to be *strongly K-nonnegative* provided that $A \overset{K}{\geqslant} 0$ and A has an eigenvector in the interior of K. We let Π^{SN} (Π_e^{SN}) denote the set of $n \times n$ matrices A for which there exists a proper (an ellipsoidal) cone K such that A is strongly K-nonnegative.

The following result may be found in [2].

THEOREM 3.5. *Let A be a real $n \times n$ matrix. Then the following are equivalent*:

$$A \in \Pi^{SN}; \tag{3.21}$$

$$\rho(A) \in \sigma(A), \text{ and } d(\lambda) = 1 \text{ for every } \lambda \in \sigma(A) \text{ such that } |\lambda| = \rho(A). \tag{3.22}$$

In fact, the following result holds.

THEOREM 3.6. *For fixed n we have*

$$\Pi^{SN} = \Pi_e^{SN}. \tag{3.23}$$

Proof. The proof is similar to that of Theorem 3.4 except that in the present situation the existence of an eigenvector in $\partial(K_n)$ is not precluded, since the real canonical form of A may have more than one 1×1 block for $\rho(A)$. In particular, if Q_2 is the 1×1 block $[\rho(A)]$, then $v = e^n + e^{n-1}$ is a boundary eigenvector. ∎

It is clear that Theorems 3.2, 3.4, and 3.6 hold for any class of proper cones which contain the ellipsoidal cones. Thus they hold for the class of rotund proper cones.

4. THE NONNEGATIVE CASE

In the previous section we saw that characterizations of properties stronger than ordinary K-nonnegativity were unchanged when we restricted ourselves to ellipsoidal cones. We shall see that this is not true for ordinary K-nonnegativity. That is, we shall show that $\Pi^N \neq \Pi_e^N$.

We first present some known results which are required.

THEOREM 4.1 ([8] and [2]). *Let A be an $n \times n$ matrix. Then the following are equivalent*:

$$A \in \Pi^N; \tag{4.1}$$

$$\rho(A) \in \sigma(A), \text{ and if } \lambda \in \sigma(A) \text{ is such that } |\lambda| = \rho(A), \text{ then } d(\lambda) \leqslant d(\rho(A)). \tag{4.2}$$

Furthermore, if $A \overset{K}{\geqslant} 0$ *for a proper cone K, then K contains an eigenvector of A corresponding to* $\rho(A)$.

THEOREM 4.2 [2]. *Let* $K \subset \mathbb{R}^n$ *be a proper cone such that* $A \overset{K}{\geqslant} 0$, *and assume that A has an eigenvector in* int K. *Then the associated eigenvalue is* $\rho(A)$ *and its degree is* 1.

We shall also need the following geometrical lemmas.

LEMMA 4.1. *Let K be an ellipsoidal cone with* $0 \neq z \in \partial K$. *Denote* $H = \{Qz\}^{\perp}$. *Then the following holds*:

(i) *H is the unique supporting hyperplane to K at z. Furthermore,* $v = Qz$ *is an* outward normal *to K at z; i.e.*

$$v^t x \leqslant 0 \qquad \textit{for all} \quad x \in K. \tag{4.3}$$

(ii) $H \cap K = \{\alpha z : \alpha \geqslant 0\}$.
(iii) $z - \epsilon v \in$ int K *for all sufficiently small* $\epsilon > 0$.

Proof. The gradient of the quadratic function $f(x) = x^t Q x$ at z is $v = 2Qz$, which is normal to the tangent plane to K at z. It is an outward normal, since f is increasing at z in the direction of its gradient. This proves (i).

In order to prove (ii), let $x \in H \cap K = H \cap \partial K$. Then

$$0 = z^t Q z = x^t Q x = z^t Q x. \tag{4.3}$$

Let $x = \sum_{i=1}^{n} \alpha_i u_i$ and $z = \sum_{i=1}^{n} \beta_i u_i$. Then (4.3) yields the following implications:

$$\sum_{i=1}^{n} \lambda_i (\beta_i)^2 = 0 \quad \Rightarrow \quad \sum_{i=1}^{n-1} \lambda_i (\beta_i)^2 = -\lambda_n (\beta_n)^2, \tag{4.4}$$

$$\sum_{i=1}^{n} \lambda_i (\alpha_i)^2 = 0 \quad \Rightarrow \quad \sum_{i=1}^{n-1} \lambda_i (\alpha_i)^2 = -\lambda_n (\alpha_n)^2, \tag{4.5}$$

$$\sum_{i=1}^{n} \lambda_i \alpha_i \beta_i = 0 \quad \Rightarrow \quad \sum_{i=1}^{n-1} \lambda_i \alpha_i \beta_i = -\lambda_n \alpha_n \beta_n. \tag{4.6}$$

Hence

$$\left(\sum_{i=1}^{n-1} \lambda_i \alpha_i \beta_i \right)^2 = \left(\sum_{i=1}^{n-1} \lambda_i (\alpha_i)^2 \right) \left(\sum_{i=1}^{n-1} \lambda_i (\beta_i)^2 \right), \tag{4.7}$$

which is equality in the Cauchy-Schwartz inequality. Thus, for some α, $\sqrt{\lambda_i}\,\alpha_i = \alpha \sqrt{\lambda_i}\,\beta_i$, $i = 1, \dots, n-1$. By (4.4) and (4.5), $\alpha_n = \pm \alpha \beta_n$, and then by (4.6) we have $\alpha_n = \alpha \beta_n$. This proves (ii).

By Taylor's theorem,

$$f(z - \epsilon v) = f(z) - \epsilon v^t v + o(\epsilon) < 0$$

for small positive ϵ. Continuity now yields (iii). ∎

LEMMA 4.2. *Let $A \overset{K}{\geqslant} 0$, where K is an ellipsoidal cone, and assume that $z \in \partial K$ is an eigenvector of A. Then $v = Qz$ is a left eigenvector of A belonging to a nonnegative eigenvalue.*

Proof. The result of Lemma 4.1(i) implies that $-Qz \in K^*$, where K^* is the dual cone

$$K^* = \{ y \in \mathbb{R}^n : y^t x \geqslant 0 \text{ for all } x \in K \}.$$

If $K = TK_n$ for a nonsingular matrix T, then since $K_n = K_n^*$ it follows that $K^* = (T^t)^{-1} K_n$, and consequently, by Proposition 2.3, K^* is ellipsoidal and therefore rotund. Then $\{z\}^{\perp}$ is the unique supporting hyperplane to K^* at $-Qz$, and therefore

$$\operatorname{span}(-Qz) = \{z\}^{\perp} \cap K^*.$$

Now, since both $\{z\}^{\perp}$ and K^* are invariant under A^t, it follows that $\operatorname{span}(-Qz)$ is invariant under A^t. Consequently, Qz is a left eigenvector of A, and the pointedness of K^* implies that the associated eigenvalue is nonnegative. ■

The next result follows from Lemma 5.3 in [7], which is itself based on results in [5]. We include a direct proof for completeness. The following terminology is adopted: Let A be a matrix in Jordan canonical form, and let B be a block corresponding to $\lambda \in \sigma(A)$. Then an eigenvector associated with λ is said to *correspond* to B provided that all its nonzero entries correspond to the position occupied by B in A. Similarly, we say that an eigenvector of a matrix not in Jordan canonical form corresponds to a Jordan block provided that the appropriately transformed eigenvector satisfies the above definition with the Jordan canonical form of A. The same terminology applies to left eigenvectors.

LEMMA 4.3. *Let* $A \overset{K}{\geqslant} 0$, *where* K *is an ellipsoidal cone. Assume that* $d(\rho(A)) > 1$. *Then the Jordan canonical form of* A *has only one maximal-order block for* $\rho(A)$. *Furthermore,* A *has only one eigenvector in* K, *this vector lies in* ∂K, *and it corresponds to the maximal-order block for* $\rho(A)$.

Proof. In view of Theorems 4.1 and 4.2, ∂K contains an eigenvector z of A corresponding to $\rho(A)$. Furthermore, K contains no other eigenvector y corresponding to $\rho(A)$, since the rotundity of K would imply that the vector $\frac{1}{2}(z+y)$, which is an eigenvector corresponding to $\rho(A)$, is in $\operatorname{int} K$.

Let v be as in Lemma 4.2. Since v is an outward normal to the rotund cone K at z, upon appropriate scaling, Lemma 4.1(iii) implies

$$z - v \in \operatorname{int} K.$$

Now let x be an eigenvector which corresponds to a maximal-order block B for $\rho(A)$. Then for an appropriately scaled left eigenvector w corresponding to B we have

$$q = z - v + w \in \operatorname{int} K.$$

There are now two cases to consider.

Case 1: $\rho(A) > 0$. In this case consider the sequence in K given by

$$\{s^k\}_{k=1}^{\infty} = \left\{ \frac{A^k q}{\|A^k q\|} \right\}_{k=1}^{\infty}.$$

This sequence has as its limit an eigenvector $u \in K$ belonging to $\rho(A)$, and unless z corresponds to B, u is not a scalar multiple of z, which is a contradiction. Furthermore, from this argument it is clear that B is unique, by considering an alternative choice for x.

Case 2: $\rho(A) = 0$. In this case, consider the vector $s = A^{d(0)-1}q$. Then $s \in K$ is an eigenvector, and unless z corresponds to B, s is not a scalar multiple of z, a contradiction. Furthermore, upon consideration of alternative x, we see that B is unique. ∎

We can now apply the above lemmas to obtain a negative example about invariant cones. (See also Lemma 5.3 below.)

EXAMPLE 4.1. Let

$$A = B(0;3) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then there does not exist an ellipsoidal cone K such that $A \overset{K}{\geqslant} 0$. Indeed, suppose that such a K did exist. Then Lemma 4.3 implies that the eigenvector $z = (0,0,\beta)^t \in \partial K$, and Lemma 4.2 implies that the outward normal $v = (\gamma,0,0)^t$ is a left eigenvector of A. Now let $\epsilon > 0$ be sufficiently small so that

$$p = z - \epsilon v \in \operatorname{int} K.$$

[Here we are applying Lemma 4.1(iii).] Then $Ap = (0, -\epsilon, 0)^t \in H \cap K$, where $H = \{v\}^{\perp}$, but $Ap \neq \alpha z$ for any $\alpha \in \mathbb{R}$. This contradicts Lemma 4.1(ii).

The above example along with Theorem 4.1 proves the following.

THEOREM 4.3. *For $n \leqslant 2$ we have $\Pi_e^N = \Pi^N$, but $\Pi_e^N \subsetneqq \Pi^N$ for $n \geqslant 3$.*

Proof. The assertion for $n = 1$ and $n = 2$ is readily verified. The rest follows from the fact that Example 4.1 generalizes in a straightforward way in case $A = B(0; n)$ for $n > 3$. ∎

The content of Theorem 4.3 is contained in our main result, Theorem 5.1 below. Note that in view of the geometric approach taken in the present section, we may replace "ellipsoidal" with "rotund" in Theorem 4.3.

5. COMPLETING THE CHARACTERIZATION OF THE NONNEGATIVE CASE

We now complete the characterization of Π_e^N in terms of spectral conditions. We first present a result on invariant ellipsoidal cones. This result was first presented by Loewy and Schneider in [4] for the special case of the ice-cream cone; see also [1]. Our extension to ellipsoidal cones follows readily from Proposition 2.3. For a proper cone K, let us denote by $\Pi(K)$ $[\tilde{\Pi}(K)]$ the set of all matrices A such that $AK \subset K$ $[AK = K]$.

LEMMA 5.1. *Let $K \subset \mathbb{R}^n$ be an ellipsoidal cone in $\mathbb{R}^n$ with an associated matrix Q, and let A be an $n \times n$ matrix. Then the following hold:*

(i) *A necessary condition for $A \in \Pi(K) \cup \{-\Pi(K)\}$ is*

$$A^t QA - \mu Q \text{ is negative semidefinite for some } \mu \geqslant 0. \tag{5.1}$$

Furthermore, this condition is sufficient in case rank $A > 1$.

(ii) *A necessary and sufficient condition for $A \in \tilde{\Pi}(K) \cup \{-\tilde{\Pi}(K)\}$ is*

$$A^t QA - \mu Q = 0 \qquad \text{for some} \quad \mu > 0. \tag{5.2}$$

We let $\tilde{\Pi}_e^N$ denote the set of $n \times n$ matrices A such that $AK = K$ for some ellipsoidal cone K.

LEMMA 5.2. *Let*

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix}, \tag{5.3}$$

where $\lambda > 0$. Then $A \in \tilde{\Pi}_e^N$.

Proof. First note that

$$A^tQA-\mu Q=0$$

if and only if

$$(\alpha A^t)Q(\alpha A)-(\alpha^2\mu)Q=0$$

for any $\alpha>0$. Hence by Lemma 5.1 and Corollary 2.1 we may assume that $\lambda=1$ in (5.3). To complete the proof, note that $\mu=1$ and

$$Q=\begin{bmatrix} 2 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{5.4}$$

satisfy (5.2), and that Q has inertia $(2,0,1)$. Let K be given by (2.2). By Lemma 5.1(ii), either $AK=K$ or $AK=-K$. But the latter case is impossible in view of Theorem 4.1 and the fact that 1 is the only eigenvalue of A. ∎

The matrix Q in (5.4) can be obtained by a heuristic method. Consider the matrix A in (5.3) with $\lambda=1$. Let $p=(1,0,1)^t$, and consider the "orbit" of p given by

$$S=\{A^sp:s\in\mathbb{R}\}.$$

Points $(x,y,z)^t\in S$ satisfy $x=1$, $y=s$, and $z=s(s-1)/2+1$ for varying values of the parameter s. Upon eliminating the parameter, we obtain the relation

$$2zx=y^2-yx+2x^2,$$

which describes a surface W in $\mathbb{R}^3$. Furthermore, it is a straightforward exercise to check that W is a nonconvex A-invariant cone. This leads naturally to the conjecture that the set

$$\{(x,y,z)^t:-2zx+y^2-yx+2x^2\leqslant 0\}$$

equals $K\cup\{-K\}$ for an ellipsoidal cone K in $\tilde{\Pi}_e^N$. Note that the associated Q describing the quadratic form is given by (5.4). Furthermore, for different vectors p one obtains different surfaces W, but the associated Q will always be of the required form. One can also determine appropriate matrices Q by this method, for positive values of λ other than 1.

REMARK 5.1. By applying Proposition 2.1 one can show that the cone K in the proof of Lemma 5.2 is actually given by

$$K = \{x \in \mathbb{R}^3 : x^t Q x \leqslant 0,\ x_3 \geqslant 0\}.$$

We shall make use of the following well-known fact.

FACT A. *If a diagonal element of a semidefinite matrix is* 0, *then the corresponding row and column must also be* 0.

The next lemma characterizes the nilpotent members of Π_e^N.

LEMMA 5.3. *Let* A *be a nilpotent* $n \times n$ *matrix. Then* A *leaves an ellipsoidal cone invariant if and only if* $d(0) \leqslant 2$ *and the Jordan canonical form of* A *has at most one* 2×2 *block.*

Proof. In view of Corollary 2.1, we can assume that A is in real canonical form. If $A = 0$, then there is nothing to prove. So we assume that $A \neq 0$. Then

$$A = \begin{bmatrix} B & 0 \\ 0 & J \end{bmatrix}, \tag{5.5}$$

where J is a largest Jordan block corresponding to $\rho(A) = 0$, and its order is greater than 1. For the case where $B = 0$ and J is 2×2, we use the matrix

$$Q = \begin{bmatrix} I & & \\ & 1 & -1 \\ & -1 & 0 \end{bmatrix}.$$

Then $u^n = (0, \dots, 1, (1+\sqrt{5})/2)^t$ and

$$K = \left\{ x = (x_1, \dots, x_n)^t : \sum_{i=1}^{n-1} x_i^2 - 2x_{n-1}x_n \leqslant 0,\ x_{n-1} + \frac{(1+\sqrt{5})x_n}{2} \geqslant 0 \right\}$$

$$= \left\{ x = (x_1, \dots, x_n)^t : \sum_{i=1}^{n-1} x_i^2 - 2x_{n-1}x_n \leqslant 0,\ x_{n-1}, x_n \geqslant 0 \right\}.$$

Now, we see that A maps K onto the ray generated by e^n, which lies in K.

If the size of J is 3 or more and a Q exists that satisfies (5.1), first note that the last column of A^tQA is 0. By Lemma 4.3, we have that the eigenvector $u = e^n$ (or $-e^n$) is in ∂K. Thus $u^tQu = 0$ and therefore $q_{nn} = 0$. If $\mu \neq 0$, we conclude that the last column of Q is 0, which contradicts the inertia requirement. If $\mu = 0$, then A^tQA must be negative semidefinite. Since Q has $n-1$ positive eigenvalues, Q is positive definite on an $(n-1)$-dimensional subspace V. Since the dimension of the range of A exceeds 1, there exists x such that $0 \neq Ax \in V$. But then $x^tA^tQAx > 0$, contradicting the negative semidefiniteness of A. The proof of the lemma is completed upon noting that A cannot have two blocks of order $2 = d(0)$. ∎

The case $n = 3$ in Lemma 5.2 is quite special, as we now show.

LEMMA 5.4. *Suppose that* $\lambda = \rho(A)$ *is an eigenvalue of an* $n \times n$ *matrix* A *where* degree $d(\lambda) \geqslant 4$. *Then* A *does not leave an ellipsoidal cone invariant.*

Proof. From Lemma 5.3, we see that we can assume $\rho(A) > 0$. Moreover, by Corollary 3.1 we can assume that $\lambda = 1$ and that A is in real canonical form; i.e.

$$A = \begin{bmatrix} B & 0 \\ 0 & J \end{bmatrix},$$

where J is a $t \times t$ Jordan block corresponding to $\lambda = \rho(A)$ and $t = d(\lambda)$. To complete the proof we need only show that no Q exists that satisfies (5.1) and has inertia $(n-1, 0, 1)$.

Suppose that such a Q (and corresponding K) did exist. Let us verify the result for $t = 4$. The general result follows similarly. Partition Q as

$$Q = \begin{bmatrix} E & F \\ F^t & H \end{bmatrix},$$

so that

$$A^tQA = \begin{bmatrix} B^tEB & B^tFJ \\ J^tF^tB & J^tHJ \end{bmatrix} \tag{5.6}$$

and

$$W = J^t H J - \mu H = \begin{bmatrix} \cdots & h_{23} + h_{14} + h_{24} & h_{24} \\ & h_{33} + h_{24} + h_{34} & h_{34} \\ & 2h_{34} + h_{44} & h_{44} \\ & h_{44} & 0 \end{bmatrix} + (1-\mu)H. \quad (5.7)$$

By Lemma 4.3, the eigenvector of A, $u = e^n$ (or $-e^n$), is in ∂K, i.e., $u^t Q u = 0$. Thus $q_{nn} = h_{44} = 0$. Since W is negative semidefinite, this implies that the last column of W is 0 by Fact A.

First consider the case $\mu = 1$. Since $h_{44} = h_{34} = h_{24} = 0$, the (3,3) element of W is 0, which implies that $h_{33} = 0$ by Fact A. Therefore the bottom right 2×2 block of Q is 0. But then $\partial K \cup \{-\partial K\}$ contains a 2-dimensional space, which violates rotundity.

Now suppose that $\mu \neq 1$. We again conclude that $h_{44} = h_{34} = h_{24} = 0$. Since the signature of Q implies that no 2×2 principal submatrix of H can be negative semidefinite (e.g. [3, Theorem 4.3.15]), we obtain $h_{33} > 0$. Moreover, since W is negative semidefinite, we have $(1-\mu)h_{33} \leqslant 0$ and so $\mu > 1$. Since the (n,n) element in $A^t Q A - \mu Q$ is 0, we conclude that the last column

$$\begin{pmatrix} (B^t - \mu I)(FJ)_4 \\ (J^t - \mu I)(HJ)_4 \end{pmatrix} = 0, \quad (5.8)$$

where $(\cdot)_4$ denotes the 4th column. Here we have used the fact that $(FJ)_4 = F_4$ and $(HJ)_4 = H_4$. Since $\mu > 1$ and $\rho(A) = 1$, both the matrices $B^t - \mu I$ and $J^t - \mu I$ are nonsingular, which means that the last column of Q is 0. This again contradicts the inertia requirement on Q. ■

LEMMA 5.5. *Suppose that the Jordan canonical form of A contains at least two blocks of order $\geqslant 2$ corresponding to eigenvalues with modulus $\rho(A)$. Then A does not leave an ellipsoidal cone invariant.*

Proof. We proceed as in the proof of Lemma 5.4. We can assume that A is in real canonical form, $\rho(A) = 1$, and

$$A = \begin{bmatrix} B & & \\ & J_2 & \\ & & J_1 \end{bmatrix},$$

where J_1 is a canonical block of maximal order ($\geqslant 2$) corresponding to the eigenvalue $\lambda_1 = 1$, and J_2 is a canonical block of maximal order ($\geqslant 2$ if λ_2 is real and $\geqslant 4$ if λ_2 is complex) corresponding to the eigenvalue $|\lambda_2| = 1$. Now suppose that Q satisfies (5.1) and has inertia $(n-1,0,1)$. We can partition Q as the symmetric block matrix

$$Q = \begin{bmatrix} E & F & G \\ F^t & H_2 & M \\ G^t & M^t & H_1 \end{bmatrix}$$

so that the block multiplication A^tQA makes sense. As above, we can use the fact that the eigenvector $u = \pm e^n \in \partial K$ in order to obtain that $u^tQu = 0$; i.e., $q_{nn} = 0$.

Let us first consider the case that $\mu \neq 1$ and proceed as in the proof of Lemma 5.4. Since λ_1 is real, the matrix $J_1^tH_1J_1 - \mu H_1$ has the same expansion as in (5.7), though it may be only of order 2. We conclude that $q_{nn} = q_{n-1,n} = 0$, $q_{n-1,n-1} > 0$, $(1-\mu)q_{n-1,n-1} \leqslant 0$, $\mu > 1$, and that the last column of Q is 0. This contradicts the inertia requirement on Q.

Now suppose that $\mu = 1$. As above, the matrix $J_1^tH_1J_1 - H_1$ has the same expansion as in (5.7) and must be negative semidefinite. As above, we have $q_{nn} = 0$, and the unit vector e^n (or $-e^n$) is in ∂K. Furthermore, the matrix J_2 contains a 2×2 principal block

$$C = \begin{bmatrix} a & 0 \\ 1 & a \end{bmatrix} \tag{5.9}$$

in its bottom right-hand corner, with $a = \pm 1$; or it contains a 4×4 principal block $C = B(a,b;4)$ defined in (3.2).

In the case that C is 2×2 and $a = \pm 1$, calculation shows that $(e^j)^t Qe^j = q_{jj} = 0$, where (j,j) is the position of the last main-diagonal entry of J_2 in A. Hence $\pm e^j \in \partial K$ and is an eigenvector of A corresponding to ± 1. The case -1 is impossible, since $AK \subset K$. The case $+1$ contradicts Lemma 4.3.

Otherwise [i.e. if $C = B(a,b;4)$], then as above, the matrix

$$C^tHC - H$$

is negative semidefinite, where H is the 4×4 block in H_2 corresponding to C. Let us show that this is impossible.

First suppose that H is positive definite with positive definite square root $D = H^{1/2}$. Then

$$T = (DCD^{-1})^t(DCD^{-1}) - I \tag{5.10}$$

is negative semidefinite. But $Z = DCD^{-1}$ is a nondiagonalizable and so nonnormal matrix with eigenvalues of modulus 1. Now Schur's inequality (see Schur [6]) states that

$$\sum_{i=1}^{4} |\lambda_i(Z)|^2 = 4 \leqslant \sum_{i=1}^{4} \sigma_i(Z)^2$$

with equality if and only if Z is normal. Here $\lambda_i(Z)$ and $\sigma_i(Z)$ are the eigenvalues and singular values of Z, respectively. Therefore $\sigma_1(Z) = \|Z\| > 1$, i.e., $\|Zx\| > 1$ for some x with unit norm. But then $x^tTx > 0$, which is a contradiction.

The inertia of Q implies that the 4×4 matrix H has at least three positive eigenvalues. Thus H cannot be negative semidefinite. If H is indefinite, then Lemma 5.1 states that $C \in \Pi(L)$ [or $-\Pi(L)$] for the ellipsoidal cone L determined by H. This contradicts the fact that $\pm 1 = \pm\rho(C) \notin \sigma(C)$. If is positive semidefinite, nonzero, and singular, then the inertia of Q implies that it must have exactly one 0 eigenvalue. Suppose $Hx = 0$. Then

$$0 \geqslant x^t(C^tHC - H)x = x^tC^tHCx \geqslant 0,$$

since H is positive semidefinite. Therefore $H(Cx) = 0$, i.e. $Cx = \alpha x$, for some real α. This contradicts the fact that C has no real eigenvalues.

Thus we have shown that no such H can exist. ∎

We can now summarize the characterizations.

THEOREM 5.1. *Let A be an $n \times n$ matrix. Then the following are equivalent*:

(i) $A \in \Pi_e^N$.

(ii) $\rho(A) \in \sigma(A)$, *and if* $\lambda \in \sigma(A)$ *is such that* $|\lambda| = \rho(A)$, *then* $d(\lambda) \leqslant d(\rho(A))$. *Furthermore*, $d(\rho(A)) \leqslant 3$ ($d(\rho(A)) \leqslant 2$ *if* $\rho(A) = 0$), *and the Jordan canonical form of A has at most one block of order* $\geqslant 2$ *corresponding to eigenvalues with modulus* $\rho(A)$.

Proof. That (i) implies (ii) follows from Theorem 4.1 and Lemmas 5.3–5.5. That (ii) implies (i) in case $\rho(A) = 0$ is settled by Lemma 5.3. Hence it remains to prove the implication in case $\rho(A) = \lambda > 0$.

First suppose that $d(\lambda) = 1$. Then from Theorem 3.6 we conclude that A leaves an ellipsoidal cone invariant.

Next suppose that $d(\lambda) = 2$. From this point on we assume that A is in real canonical form. Then A is of the form

$$A = \begin{bmatrix} B & \\ & B(\lambda;2) \end{bmatrix},$$

and we take

$$Q = \begin{bmatrix} D^2 & & \\ & 0 & -1 \\ & -1 & 0 \end{bmatrix},$$

where D is a diagonal positive definite matrix such that $B^tD^2B - \lambda^2D^2$ is negative semidefinite. This is equivalent to $\|DBD^{-1}\| \leqslant \lambda$. Note that such a D, with $\|D\| = 1$, always exists. This can be seen as follows. We can choose the diagonal element $d_{ii} = 1$ if the corresponding block in B is 1×1. Otherwise, for a block $B(a, b; 2k)$ of B, choose the corresponding block of D to be $D_\epsilon = \operatorname{diag}(1, 1, \epsilon, \epsilon, \epsilon^2, \epsilon^2, \dots, \epsilon^{k-1}, \epsilon^{k-1})$. Then $D_\epsilon B(a, b; 2k) D_\epsilon^{-1}$ has main-diagonal blocks all equal to $B(a, b; 2)$, subdiagonal blocks all equal to ϵI_2, and zero elsewhere. As $\epsilon \to 0$, the matrix $D_\epsilon B(a, b; 2k) D_\epsilon^{-1}$ tends to $\operatorname{diag}(B(a, b; 2), \dots, B(a, b; 2))$ (k times), which has norm $a^2 + b^2 < \lambda$. Hence one can choose a positive ϵ such that $\|D_\epsilon B(a, b; 2k) D_\epsilon^{-1}\| < \lambda$. For a block $B(\alpha; k)$ of B (where α is real), choose the corresponding block of D to be $D_\epsilon = \operatorname{diag}(1, \epsilon, \epsilon^2, \dots, \epsilon^{k-1})$, and a similar argument applies. (If $n = 2$, then $A = B(\lambda; 2)$ and we take

$$Q = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

instead.) Now consider the cone

$$K = \{x \in \mathbb{R}^n : x^tQx \leqslant 0,\ x_n \geqslant 0\}. \tag{5.11}$$

It is easily checked, using Proposition 2.1, that K is ellipsoidal, since Q has the correct inertia. Furthermore, rank $A > 1$ and by direct calculation

$$A^tQA - \mu Q = \begin{bmatrix} B^tD^2B - \mu D^2 & \\ & \begin{bmatrix} -2\lambda & -\lambda^2+\mu \\ -\lambda^2+\mu & 0 \end{bmatrix} \end{bmatrix}.$$

This matrix is negative semidefinite if and only if $\mu = \lambda^2$. Thus $AK \subset K$ by Lemma 5.1. Theorem 4.1 is needed here to rule out the possibility that $-AK \subset K$.

Finally, suppose that $d(\lambda) = 3$, and let us assume that $\lambda = 1$. Then

$$A = \begin{bmatrix} B & \\ & B(\lambda;3) \end{bmatrix}.$$

In this case we take

$$Q = \begin{bmatrix} D^2 & \\ & \tilde{Q} \end{bmatrix},$$

where $\tilde{Q}$ is the matrix given in (5.4) and D is as above. (If $n = 3$, we have $A = B(\lambda;3)$ and we simply take $Q = \tilde{Q}$.) Then Q has inertia $(n-1,0,1)$, and by Proposition 2.1, the cone

$$K = \{x \in \mathbb{R}^n : x^tQx \leqslant 0,\ x_n \geqslant 0\}$$

is an ellipsoidal cone. As above, we calculate that

$$A^tQA - \mu Q = \begin{bmatrix} b^tD^2B - \mu D^2 & \\ & \begin{bmatrix} 2\lambda^2-\lambda+1-2\mu & -\frac{1}{2}\lambda^2+\frac{1}{2}\mu & -\lambda^2+\mu \\ -\frac{1}{2}\lambda^2+\frac{1}{2}\mu & \lambda^2-\mu & 0 \\ -\lambda^2+\mu & 0 & 0 \end{bmatrix} \end{bmatrix},$$

which is negative semidefinite if and only if $\mu = \lambda^2$. Since rank $A > 1$, we again conclude that $AK \subset K$ by Lemma 5.1. ■

An interesting open problem remains to be answered, that is, can "ellipsoidal" be replaced by "rotund" in Theorem 5.1?

The argument in the above proof can also be used to deduce Theorems 3.2, 3.4, and 3.6. Assuming that $\rho(A) = 1$, and that A is in real canonical form with the 1×1 block [1] at its bottom right corner, choose the matrix Q to be $D^2\oplus[-1]$, where D is a positive diagonal matrix such that $B^tD^2B-D^2$ is negative semidefinite with B equal to the direct sum of all blocks of A except its last block. In Theorem 3.4, to show the irreducibility of A with respect to K, note that $\pm e^n\in \operatorname{int} K$ is an eigenvector corresponding to 1, and all eigenvectors corresponding to real eigenvalues other than 1 lie in $\{e^n\}^{\perp}$, which meets K only at the zero vector.

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REFERENCES

1 D. Z. Dokovic, Quadratic cones invariant under some linear operators, *SIAM J. Algebraic Discrete Methods* 8:186–191 (1987).
2 L. Elsner, Monotonie und Randspektrum bei vollstetigen Operatoren, *Arch. Rational Mech. Anal.* 36:356–365 (1970).
3 R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge U.P., 1985.
4 R. Loewy and H. Schneider, Positive operators on the n-dimensional ice-cream cone, *J. Math. Anal. Appl.* 49:375–392 (1975).
5 H. Schneider, Geometric conditions for the existence of positive eigenvalues of matrices, *Linear Algebra Appl.* 38:253–271 (1981).
6 I. Schur, Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen, *Math. Ann.* 66:488–510 (1909).
7 B. S. Tam, On the distinguished eigenvalues of a cone preserving map, *Linear Algebra Appl.*, to appear.
8 J. S. Vandergraft, Spectral properties of matrices which have invariant cones, *SIAM J. Appl. Math.* 16:1208–1222 (1968).

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