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EXPONENTIAL NONNEGATIVITY ON THE ICE CREAM CONE*

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Abstract. Let K_n denote the n-dimensional ice cream cone. This paper investigates the structure of those matrices A such that $e^{tA}K_n \subset K_n$ for all $t \geqq 0$. The characterizations extend to general ellipsoidal cones.

Key words. ice cream cone, ellipsoidal cone, matrices, exponential nonnegativity, copositivity, spectrum

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1. Introduction. A set $C \subset R^n$ is a *cone* provided that $\alpha C \subset C$ for all $\alpha \geqq 0$. We call a cone C *proper* provided that it is closed, convex, possesses nonempty interior, and is pointed ($C \cap \{-C\} = \{0\}$). Given a proper cone $C \subset R^n$, we denote by $p(C)$ the set of matrices $A \in R^{n,n}$ which are *exponentially nonnegative* on C; that is, $e^{tA}C \subset C$ for all $t \geqq 0$, where $e^{tA} = \sum_{j=0}^{\infty} (tA)^j/j!$ is the familiar matrix exponential. Hence $p(C)$ is the set of matrices A such that for an arbitrary start point $x(0) \in C$, the solution $x(t) = e^{tA}x(0)$ of the linear differential equation $\dot{x}(t) = Ax(t)$ remains in C for all future time.

The purpose of this paper is to investigate the structure of the set of matrices $p(K_n)$, where

$$K_n = \left\{ x \in R^n : \sum_{i=1}^{n-1} x_i^2 \leqq x_n^2,\ x_n \geqq 0 \right\}$$

is the *n-dimensional ice cream cone.* It will be seen that our results can be extended to general ellipsoidal cones.

In the following section, we review some required technical material on ellipsoidal cones. Then, in § 3, the main results are presented. A key result which we employ is a lemma on copositivity for the ice cream cone K_n due to Loewy and Schneider [3]. To a certain extent our results complement some of those in [3], which provided characterizations of those matrices which leave K_n invariant.

2. Ellipsoidal cones. Let $Q \subset R^{n,n}$ be a symmetric nonsingular matrix, with a single negative eigenvalue λ_n. Therefore Q has *inertia* $(n-1, 0, 1)$, where by inertia we mean the triple (P, Z, N), indicating the number of positive, zero, and negative eigenvalues, respectively. Let u_n be a unit eigenvector of Q corresponding to λ_n. With Q we associate two *ellipsoidal cones*; these are

$$K = K(Q, u_n) = \{x \in R^n : x^tQx \leqq 0,\ x^tu_n \geqq 0\} \tag{2.1}$$

and $-K = K(Q, -u_n)$. In the sequel we will employ the fact that at each $0 \neq x \in \partial K = \{x \in K : x^tQx = 0\}$, the vector Qx is an outward pointing normal at x (where ∂ denotes boundary).

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Clearly, K_n is an ellipsoidal cone with

$$Q = Q_n := \left(\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & -1 \end{array} \right) \quad \text{and} \quad u_n = e_n,$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. Also, we denote the kth unit vector by e_k.

We shall require the following lemma from [5], which says that in formula (2.1) we may replace the eigenvector u_n with vectors v satisfying certain requirements (which are met by u_n itself).

LEMMA 2.2. *Suppose that K is as above and assume that $v \in R^n$ satisfies*

$$\{v\}^{\perp} \cap \{K \cup \{-K\}\} = \{0\} \tag{2.3}$$

and

$$v^t u_n \geqq 0. \tag{2.4}$$

Then

$$K = \{x \in R^n : x^t Q x \leqq 0,\ x^t v \geqq 0\}. \tag{2.5}$$

Remark 2.6. In view of the fact that the orthogonal complement $\{u_n\}^{\perp}$ is a hyperplane which supports the proper cones K and $-K$ only at the origin, it follows from the preceding lemma that if v is a vector whose distance from u_n is sufficiently small, then (2.5) holds.

For Q as above, let the spectrum be $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ where $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{n-1} > 0 > \lambda_n$, and let the orthogonal diagonalization of Q be given by $U^tQU = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. The following lemma will also prove to be useful. Its proof, which employs Sylvester's theorem, may be found in [5].

LEMMA 2.7. *K is an ellipsoidal cone as in* (2.1) *if and only if $K = TK_n$ for some nonsingular $T \in R^{n,n}$.*

In particular, for a given ellipsoidal cone $K = K(Q, u_n)$, we have $K = TK_n$ for $T = UD$, where D is the diagonal matrix with entries $d_{ii} = |\lambda_i|^{-1/2}$, $i = 1, 2, \cdots, n$, and then $Q = (T^{-1})^t Q_n T^{-1}$. Conversely, for a given nonsingular $T \in R^{n,n}$, the matrix $(T^{-1})^t Q_n T^{-1}$ has inertia $(n-1, 0, 1)$ and $TK_n = K((T^{-1})^t Q_n T, (T^{-1})^t e_n)$.

3. Main results. To begin, we require the following lemma, in which $\langle \cdot, \cdot \rangle$ denotes the standard inner product on R^n.

LEMMA 3.1. *Let K be an ellipsoidal cone as in* (2.1). *Then*

$$p(K) = \{A \in R^{n,n} : \langle Ax, Qx \rangle \leqq 0 \text{ for all } x \in \partial K\}. \tag{3.2}$$

Proof. Since Qx is the unique outward pointing normal vector (up to scalar multiples) to K at any nonzero $x \in \partial K$, then the condition that $\langle Ax, Qx \rangle \leqq 0$, for all such x, is, in the terminology of Schneider and Vidyasagar [4], *cross-positivity* of A on K, which was shown in [4] to be equivalent to exponential nonnegativity. □

We now turn our attention to the problem of characterizing $p(K_n)$. We will make use of the following copositivity result from Loewy and Schneider [3].

LEMMA 3.3 [3, Lemma 2.2]. *Let $W \in R^{n,n}$ be symmetric. Then there exists $\mu \geqq 0$ such that $W - \mu Q_n$ is negative semidefinite if and only if*

$$x \in K_n \Rightarrow x^t W x \leqq 0. \tag{3.4}$$

Our main characterization of $p(K_n)$ is given next.

THEOREM 3.5. *A necessary and sufficient condition for $A \in p(K_n)$ is that there exists $\xi \in R$ such that*

$$Q_nA + A^tQ_n - \xi Q_n \leqq 0, \tag{3.6}$$

where "$\leqq$" means negative semidefinite.

Proof. Let us denote

$$W(Q_n, A) := Q_nA + A^tQ_n.$$

Upon symmetrizing the quadratic form $\langle Ax, Qx\rangle$, it follows that $A \in p(K_n)$ if and only if

$$x \in \partial K_n \Rightarrow x^tW(Q_n, A)x \leqq 0. \tag{3.7}$$

Since $x^tQ_nx = 0$ for all $x \in \partial K_n$, we have that (3.7) is equivalent to

$$x \in \partial K_n \Rightarrow xW(Q_n, A + \gamma I)x \leqq 0 \tag{3.8}$$

for any given $\gamma \in R$. Since

$$W(Q_n, A + \gamma I) = W(Q_n, A) + 2\gamma Q_n, \tag{3.9}$$

we may choose γ large enough to ensure that $W(Q_n, A + \gamma I)$ has inertia $(n-1, 0, 1)$. For such γ, consider the ellipsoidal cone

$$C(\gamma) := \{x \in R^n : x^tW(Q_n, A + \gamma I)x \leqq 0,\ x^tu_n(\gamma) \geqq 0\},$$

where $u_n(\gamma)$ is a unit eigenvector of $W(Q_n, A + \gamma I)$ corresponding to its only negative eigenvalue. Since γ may be chosen so large that $u_n(\gamma)$ approximates e_n to any prescribed tolerance, Remark 2.6 tells us that for sufficiently large γ we have

$$C(\gamma) = \{x \in R^n : x^tW(Q_n, A + \gamma I)x \leqq 0,\ x^te_n \geqq 0\}. \tag{3.10}$$

Hence (3.8) implies that $A \in p(K_n)$ if and only if for all γ sufficiently large we have

$$\partial K_n \subset C(\gamma). \tag{3.11}$$

Since $C(\gamma)$ is an ellipsoidal and therefore convex cone for large γ, it follows that for such γ, (3.11) is equivalent to

$$K_n \subset C(\gamma). \tag{3.12}$$

Therefore, Lemma 3.3 implies that $A \in p(K_n)$ if and only if for each sufficiently large γ there exists $\mu_\gamma \geqq 0$ such that

$$W(Q_n, A + \gamma Q) - \mu_\gamma Q_n \leqq 0. \tag{3.13}$$

Since

$$W(Q_n, A + \gamma I) - \mu_\gamma Q_n = W(Q_n, A) + (2\gamma - \mu_\gamma)Q_n, \tag{3.14}$$

the theorem is proven. □

In what follows, we shall partition A as

$$A = \left(\begin{array}{c|c} A_1 & c \\ \hline d^t & a_{nn} \end{array}\right),$$

where A_1 denotes the leading $(n-1) \times (n-1)$ principal submatrix of A. Then

$$W(Q_n, A) = \left(\begin{array}{c|c} A_1 + A_1^t & g \\ \hline g^t & -2a_{nn} \end{array} \right), \tag{3.15}$$

where

$$g := c - d,$$

and therefore

$$W(Q_n, A) - \xi Q_n = \left(\begin{array}{c|c} A_1 + A_1^t - \xi I_{n-1} & g \\ \hline g^t & \xi - 2a_{nn} \end{array} \right). \tag{3.16}$$

We have the following corollary to Theorem 3.5. It provides sufficient conditions for membership and nonmembership in $p(K_n)$.

COROLLARY 3.17. *Let $A \in R^{n,n}$. Then the following hold*:

$$\max_{1 \le i \le n-1} \left\{ 2a_{ii} + |g_i| + \sum_{i \ne j = 1}^{n-1} |a_{ij} + a_{ji}| \right\} \le 2a_{nn} - \sum_{i=1}^{n-1} |g_i| \Rightarrow A \in p(K_n), \tag{3.18}$$

$$\max_{1 \le i \le n-1} \left\{ 2a_{ii} - |g_i| - \sum_{i \ne j = 1}^{n-1} |a_{ij} + a_{ji}| \right\} > 2a_{nn} + \sum_{i=1}^{n-1} |g_i| \Rightarrow A \notin p(K_n). \tag{3.19}$$

Proof. Theorem 3.5 implies that $A \in p(K_n)$ if and only if there exists $\xi \in R$ such that the (symmetric) matrix $W(Q_n, A) - \xi Q_n$ has no positive eigenvalues. A straightforward application of Gershgorin's theorem then yields (3.18) and (3.19). □

A different sufficient condition for $A \in p(K_n)$ is provided in the following result. We shall denote the euclidean norm by $\|\cdot\|$, and the largest eigenvalue of a symmetric matrix M by $\lambda_1(M)$.

THEOREM 3.20. *A sufficient condition for $A \in p(K_n)$ is*

$$\lambda_1(A_1 + A_1^t) \le 2(a_{nn} - \|g\|). \tag{3.21}$$

Proof. Let us write

$$W(Q_n, A) - \xi Q_n = U(\xi) + V,$$

where

$$U(\xi) = \left(\begin{array}{c|c} A_1 + A_1^t - \xi I_{n-1} & 0 \\ \hline 0 & \xi - 2a_{nn} \end{array} \right) \quad \text{and} \quad V = \left(\begin{array}{c|c} 0 & g \\ \hline g^t & 0 \end{array} \right).$$

Then, since $U(\xi)$ and V are symmetric, we have

$$\lambda_1(U(\xi) + V) \le \lambda_1(U(\xi)) + \lambda_1(V). \tag{3.22}$$

(See, e.g., Wilkinson [6, p. 101].) Therefore, in view of Theorem 3.5, a sufficient condition for $A \in p(K_n)$ is the existence of $\xi \in R$ such that

$$\lambda_1(U(\xi)) + \lambda_1(V) \le 0. \tag{3.23}$$

Since $\lambda_1(V) = \|g\|$, the existence of such a ξ is readily seen to be guaranteed by (3.21). □

It is not difficult to construct examples where the sufficient condition (3.21) holds, but (3.18) fails. The reverse may occur as well, as is evidenced by the matrix

$$A=\begin{pmatrix} -2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The next result provides a general necessary condition for $A \in p(K_n)$.

THEOREM 3.24. *Let $A \in p(K_n)$. Then*

$$\lambda_1(A_1 + A_1^t) \leqq 2a_{nn}. \tag{3.25}$$

Proof. Theorem 3.5 tells us that if $A \in p(K_n)$, then there exists a real number ξ such that all the spectrum of $W(Q_n, A) - \xi Q_n$ is nonpositive, which implies that each principal submatrix has nonpositive spectrum as well. Applying this fact to the principal submatrices $A_1 + A_1^t - \xi I_{n-1}$ and $\xi - 2a_{nn}$ readily yields (3.25). □

Theorems 3.20 and 3.24 immediately yield the following complete characterization of $p(K_n)$ for matrices satisfying a certain "partial symmetry" condition.

COROLLARY 3.26. *Let $A \in R^{n,n}$ be such that $a_{in} = a_{ni}$ for all $1 \leqq i \leqq n-1$ (i.e., $g = 0$). Then (3.25) is necessary and sufficient for $A \in p(K_n)$.*

Another general necessary condition is given next.

THEOREM 3.27. *Assume that $A \in p(K_n)$. Let $\{\mu_1, \mu_2, \cdots, \mu_k\}$ be any set of eigenvalues of A (not necessarily distinct), and let $\{x_1, x_2, \cdots, x_k\}$ be a corresponding set of eigenvectors. Consider the (possibly empty) index sets*

$$I_+ = \{i : x_i^* Q_n x_i > 0\} \quad \text{and} \quad I_- = \{i : x_i^* Q_n x_i < 0\}.$$

Then

$$\inf\{\text{Re}\,\mu_i : i \in I_-\} \geqq \sup\{\text{Re}\,\mu_i : i \in I_+\} \tag{3.28}$$

(where $\sup(\varnothing) = -\infty$ and $\inf(\varnothing) = \infty$, $\varnothing$ denoting the empty set).

Proof. Since $A \in p(K_n)$, there exists $\xi \in R$ such that

$$H(\xi) := Q_n A + A^t Q_n - \xi Q_n \leqq 0. \tag{3.29}$$

Then

$$x_i^* H(\xi) x_i = 2x_i^* Q_n x_i (\text{Re}\,\mu_i - \xi) \leqq 0 \quad \text{for all } i = 1, 2, \cdots, k. \tag{3.30}$$

Hence $\xi \geqq \text{Re}\,\mu_i$ for all $i \in I_+$ and $\xi \leqq \text{Re}\,\mu_i$ for all $i \in I_-$, yielding (3.28). □

Our final result provides a characterization of the set of matrices

$$p(\partial K_n) := \{A \in R^{n,n} : e^{tA}(\partial K_n) \subset \partial K_n \text{ for all } t \geqq 0\}.$$

Hence $p(\partial K_n)$ is the set of matrices A such that solutions of the linear differential equation $\dot{x}(t) = Ax(t)$ with $x(0) \in \partial K_n$ remain in ∂K_n for all $t \geqq 0$.

THEOREM 3.31. *A necessary and sufficient condition for $A \in p(\partial K_n)$ is that $A = B + \alpha I$, where $\alpha \in R$ and*

$$B = \left(\begin{array}{c|c} B_1 & b \\ \hline b^t & 0 \end{array}\right)$$

with B_i being an $(n-1) \times (n-1)$ skew-symmetric matrix.

Proof. The matrix $A \in p(\partial K_n)$ if and only if the vector field Ax is tangent to the locally smooth surface $\partial K_n / \{0\}$; that is,

$$\langle Ax, Qx \rangle = 0 \quad \text{for all } x \in \partial K_n. \tag{3.32}$$

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This is equivalent to $A \in p(K_n)$ and $-A \in p(K_n)$. Hence in view of Theorem 3.5, (3.32) is equivalent to the existence of real numbers ξ_1 and ξ_2 such that

$$(3.33) \qquad W(Q_n, A) - \xi_1 Q_n \leqq 0 \quad \text{and} \quad W(Q_n, -A) - \xi_2 Q_n \leqq 0.$$

But (3.33) implies that $\xi_1 = -\xi_2$ and $W(Q_n, A) = \xi_1 Q_n$. In view of (3.15), the conclusion of the theorem follows. □

We conclude with some remarks.

Remark 3.34. (i) The proof of Theorem 3.31 shows that $p(\partial K_n)$ is the maximal subspace of the closed convex cone $p(K_n) \in R^{n,n}$. The theorem implies that $\dim(p(\partial K_n)) = (n^2 - n + 2)/2$.

(ii) It is interesting to note that if A satisfies either of the sufficient conditions (3.18) or (3.21), or if A is of the form specified in Theorem 3.31, then A must satisfy the conditions of Elsner [1] for the existence of a proper cone K such that $A \in p(K)$; namely, that the *spectral abscissa*

$$\lambda(A) := \max\{\operatorname{Re}\lambda : \lambda \text{ is an eigenvalue of } A\}$$

is an eigenvalue of A and no eigenvalue λ of A with $\operatorname{Re}\lambda = \lambda(A)$ can have degree exceeding that of $\lambda(A)$. (By the degree of an eigenvalue, we mean its degree in the minimal polynomial.)

(iii) Our results can be extended to general ellipsoidal cones by applying Lemma 2.7. In particular, let $K = K(Q, u_n)$ be a given ellipsoidal cone, and let T be a nonsingular matrix such that $K = TK_n$. (One such T is provided by Lemma 2.7.) Then $A \in p(K)$ if and only if $T^{-1}AT \in p(K_n)$, and likewise, $A \in p(\partial K)$ if and only if $T^{-1}AT \in p(\partial K_n)$.

(iv) In view of (3.7), $A \in p(K_n)$ if and only if $x^t W(Q_n, A)x \leqq 0$ for all $x \in R^n$ such that $x_n = 1$ and $\sum_{i=1}^{n-1} x_i^2 = 1$. Hence a necessary and sufficient condition for $A \in p(K_n)$ is

$$(3.35) \qquad \max\{y^t(A_1 + A_1^t)y + 2y^t(c-d) : \|y\| = 1\} \leqq 0.$$

A numerical method for obtaining the maximum in (3.35) may be found, e.g., in Fletcher [2]. Thus we can computationally check whether $A \in p(K_n)$ in cases where our necessary conditions are met, but sufficiency is not.

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