Mathematical Programming 57 (1992) 85-101
North-Holland

Generalizations of Slater's constraint qualification for infinite convex programs

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Received 31 May 1988
Revised manuscript received 12 June 1990

In this paper we study constraint qualifications and duality results for infinite convex programs

(P) $\mu = \inf\{f(x)\colon g(x) \in -S,\ x \in C\}$,

where $g = (g_1, g_2)$ and $S = S_1 \times S_2$, S_i are convex cones, $i = 1, 2$, C is a convex subset of a vector space X, and f and g_i are, respectively, convex and S_i-convex, $i = 1, 2$. In particular, we consider the special case when S_2 is in a *finite* dimensional space, g_2 is affine and S_2 is polyhedral. We show that a recently introduced simple constraint qualification, and the so-called quasi relative interior constraint qualification both extend to (P), from the special case that $g = g_2$ is affine and $S = S_2$ is polyhedral in a finite dimensional space (the so-called partially finite program). This provides generalized Slater type conditions for (P) which are much weaker than the standard Slater condition. We exhibit the relationship between these two constraint qualifications and show how to replace the affine assumption on g_2 and the finite dimensionality assumption on S_2, by a local compactness assumption. We then introduce the notion of strong quasi relative interior to get parallel results for more general infinite dimensional programs without the local compactness assumption. Our basic tool reduces to guaranteeing the closure of the sum of two closed convex cones.

Key words: Constraint qualifications, characterization of optimum, duality, convex programming, constrained best approximation, partially finite programs, infinite programs, closure of sums of cones.

1. Introduction

In this paper we study constraint qualifications and duality results for the abstract convex program (P). We show that the constraint qualifications, introduced in [4, 5] for partially finite programs (P), where g is affine and S is polyhedral and finite dimensional, can be extended to more general infinite convex programs. We show the relationships between the two constraint qualifications as well as their relation to several new ones. In particular, we demonstrate two methods to remove the finite dimensional assumption. Throughout, our technique is to reduce the problem to that of guaranteeing the closure of the sum of two closed convex cones.

The concept of quasi relative interior (qri) is introduced in [4] for partially finite programs (P). This generalizes the notion of relative interior (ri) to infinite dimensions. It allows a corresponding weakened generalized Slater condition for (P), i.e., we essentially need only to find a feasible point $\hat{x}$ in the qri of C for which the nonaffine part of g at $\hat{x}$ lies in the interior (int) of the corresponding part of $-S$. If g is affine, then this reduces to finding a feasible point in the qri of C, which is the constraint qualification given in [4]. Since the generalized Slater condition fails frequently in infinite dimensions, i.e., the ri of a convex set is often empty, this substantially strengthens optimality results.

The dual approach to the constraint qualification is provided by looking at polar cones and supporting hyperplanes. This is the approach used in [5], where the simple constraint qualification is essentially equivalent to a statement about supporting hyperplanes to C which contain the feasible set, but do not contain all of C.

In [4], the notion of qri is studied extensively. For example, it is shown that: qri coincides with ri in finite dimensional normed spaces; qri is convex if C is convex; and qri $C \neq \emptyset$ if C is a convex subset of X and X is the dual of a normed space and has the w^*-topology. (We include several of these properties in Section 3, for the convenience of the reader, along with other needed derived results.)

It is also shown in [4] that the partially finite programs have wide applications such as in best approximation problems and semi-infinite programming. The importance of qri is seen by the fact that the qri constraint qualification can be easily checked in many situations by just checking feasibility. Many sets which have empty ri are seen to have nonempty qri.

The qri in [4] and the simple constraint qualification in [5] (along with modifications in this paper) provide constraint qualifications for partially finite programs, i.e., programs with affine constraints and finite dimensional range. We show that the important property that makes these constraint qualifications work is not the affineness or finite dimensionality, but rather it is the local compactness which arises in these situations. We also consider the notion of strong quasi relative interior (sqri) and show that it provides a constraint qualification when paired with a finite dimensionality assumption in the domain space. This parallels the partially finite case where the finite dimensional assumption is made in the range space. We see that the sqri is equivalent to ri for closed convex sets and also, we show that it provides a generalization of the classical interiority range constraint qualification.

In Section 2 we provide the preliminary definitions and results used in the paper. We also include several conditions which guarantee that the sum of two closed convex cones is closed. The constraint qualifications are presented in Section 3. Sections 4 and 5 present the main duality (Karush–Kuhn–Tucker conditions) theorems for (P). We provide proofs based on the closure of the sum of two cones and show that nonclosure of the sum is an alternative to the constraint qualification. The results in these sections apply to the partially finite programs (P) and are extended to more general infiite programs using a locally compact assumption and the notion of strong quasi relative interior. We conclude in Section 6 with a discussion on the various constraint qualifications presented in the paper, as well as some directions for future research.

2. Preliminaries and the closure of the sum of cones

Consider the *abstract convex program*

(P) $\mu = \inf f(x)$

subject to $g(x) \in -S$,

$x \in C$,

where $f: X \to \mathbb{R}$, $g: X \to Y$; X and Y are real Banach spaces; $C \subset X$ and $S \subset Y$ are convex and moreover, S is a cone, i.e., $\lambda s \in S$, for all $\lambda \geq 0$ and all $s \in S$; f is a continuous convex functional and g is continuous and *S-convex*, i.e.,

$$tg(x_1) + (1-t)g(x_2) - g(tx_1 + (1-t)x_2) \in S, \tag{2.1}$$

for all x_1, x_2 and all t in $[0, 1]$. We let $f^0(x, h)$ denote the directional derivative of f in the direction d, and ∂f denote the subdifferential of f, see e.g. [9, 17].

We further assume that $g = (g_1, g_2)$ and

$$S = S_1 \times S_2 \subset Y = Y_1 \times Y_2.$$

In many of the results, we will assume that Y_2 is finite dimensional, g_2 is an affine function and S_2 is a polyhedral cone. Thus

$$g_2(x) = Ax - b \quad \text{for some linear } A: X \to Y_2,\ b \in Y_2,$$

$$S_2 = \{y \in Y_2 : By \geq 0\} \quad \text{for some matrix } B.$$

If, in addition to these assumptions, $g = g_2$, then (P) is called a *partially finite program*, see [4].

We let

$$D = \{x \in X : g_2(x) \in -S_2\}$$

and

$$E = \{x \in X : g_1(x) \in -S_1\}.$$

The cone S induces a *partial order on Y* given by

$$x_1 \geq_S x_2 \Leftrightarrow x_1 - x_2 \in S.$$

We let

$$F = g^{-1}(-S) \cap C = C \cap D \cap E$$

denote the *feasible set* of (P).

We say that x is a *nonsupport point* of C if every closed supporting hyperplane to C at x contains C. We say that a hyperplane *properly supports* C if it supports C and does not contain all of C.

K is a *face* of a convex set C if K is a convex subset of C, and, whenever $c_1, c_2 \in C$, then

$$\tfrac{1}{2}(c_1 + c_2) \in K \;\Rightarrow\; c_1, c_2 \in K. \tag{2.2}$$

A face K is *exposed* if there exists $\bar{c} \in K$ and $\varphi \in (C - \bar{c})^+$ such that

$$K = \{c \in C : \varphi(c - \bar{c}) = 0\}. \tag{2.3}$$

Note that for any set K in X, we let

$$K^+ = \{\varphi \in X^* : \varphi x \geq 0 \;\forall x \in K\}$$

denote the *polar cone* of K and $K^- = -K^+$, where X^* denotes the topological dual space of X equipped with the w^*-topology. The *annihilator* of K is $K^\perp = K^+ \cap K^-$. Correspondingly, if $L \subset X^*$, then

$$L^+ = \{x \in X : \varphi x \geq 0 \;\forall \varphi \in L\}. \tag{2.4}$$

The cones L^- and $L^\perp$ are defined similarly. Note that

$$K^{++} = \overline{\text{cone}}\, K,$$

where $\overline{\text{cone}}$ denotes the closure of the generated convex cone. Moreover, if K, L are closed convex cones, then

$$(K \cap L)^+ = \overline{K^+ + L^+}. \tag{2.5}$$

We shall need conditions that guarantee that the sum of two closed cones is closed. Two well known conditions are given in the following lemma (see e.g., [11, Section 15.D]):

Lemma 2.1. (a) *If K, L are convex sets and $0 \in K \cap L$, then*

$$K \cap \text{int}\, L \neq \emptyset \quad \textit{implies} \quad (K \cap L)^+ = K^+ + L^+.$$

(b) *If K and L are closed convex cones, then*

$$K \cap (-L) = \{0\},\ K \textit{ locally compact} \quad \textit{implies} \quad K + L \textit{ closed.} \qquad \square$$

Three new conditions are:

Lemma 2.2. (a) *Suppose that K and L are closed convex sets, $0 \in K \cap L$ and*

$$\text{cone}(K - L) \textit{ is a closed subspace.} \tag{2.6}$$

Then

$$(K \cap L)^+ = K^+ + L^+.$$

(b) *Suppose that X is a Hilbert space, K and L are closed convex cones, and the angle between K and $-L$ is >0, i.e.,*

$$\inf\{kl : \|k\| = \|l\| = 1,\ k \in K,\ l \in L\} < 1.$$

Then $K + L$ is closed.

(c) *Suppose that C is a closed subspace and D is a finite dimensional subspace. Then $C + D$ is closed.*

Proof. (a) Let φ and ψ be the indicator functions for the sets K and L, respectively. Then they are lower semi-continuous convex functions with domains K and L, respectively. Also, the subdifferential

$$\begin{aligned}\partial\varphi(0) &= \{v \in X^*: vx \leq -\varphi(0)+\varphi(x)\ \forall x \in X\}\\ &= \{v \in X^*: vx \leq \varphi(x)\ \forall x \in X\}\\ &= \{v \in X^*: vx \leq 0\ \forall x \in K\}\\ &= -K^+.\end{aligned}$$

Similarly, $\partial\psi(0) = -L^+$ and $\partial(\varphi+\psi)(0) = -(K \cap L)^+$. The result follows from the theorem in [1] which states that

$$\partial(\varphi+\psi)(x) = \partial\varphi(x) + \partial\psi(x) \quad \forall x \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi, \tag{2.7}$$

if

$$\operatorname{cone}(\operatorname{dom}\varphi - \operatorname{dom}\psi) \text{ is a closed subspace,} \tag{2.8}$$

where dom φ denotes the domain of φ.

(b) Suppose that $k_n \in K$, $l_n \in L$, and $k_n + l_n = z_n \to z \notin K+L$. Since K and L are closed, we can assume that both sequences are unbounded. In fact, we can assume that $\|k_n\| \to \infty$. Therefore,

$$\lim k_n/\|l_n\| = \lim(z_n - l_n)/\|l_n\| = \lim -l_n/\|l_n\|,$$

which implies that the angle between k_n and $-l_n$ approaches zero, a contradiction.

(c) By decomposing D into the sum of two subspaces, we can assume that $C \cap D = \{0\}$. The result now follows from Lemma 2.1(b). □

We will also need to relate the subdifferentials and the cones of feasible directions.

Lemma 2.3. *Suppose that* $g(\hat{x}) \in -\operatorname{int} S$ *and* $g(x^*) \in -S$. *Let*

$$G = \{x: g(x) \in -S\}.$$

Then

$$\begin{aligned}(G-x^*)^- &= \Big(\bigcap_{\lambda \in S^+,\, \lambda g(x^*)=0} \{d: (\lambda g)^0(x^*, d) \leq 0\}\Big)^-\\ &= \operatorname{cone}\{\varphi: \varphi \in \partial(\lambda g)(x^*) \textit{ for some } \lambda \in S^+ \textit{ with } \lambda g(x^*) = 0\}.\end{aligned} \tag{2.9}$$

Proof. Let $\mathbf{P}_S$ be a compact, convex, generating set (base) for the intersection of S^+ with $\{\lambda: \lambda g(x^*) = 0\}$. Note that $0 \notin \mathbf{P}_S$. The existence of the base is guaranteed by the fact that the interior of S is nonempty, see e.g. [7, Lemma 2.3]. Since the directional derivative is positively homogeneous, we can take the intersection in (2.9) over the base $\mathbf{P}_S$. Then the right-hand side of (2.9) becomes

$$\begin{aligned}\Big(\bigcap_{\lambda \in \mathbf{P}_S} (\partial(\lambda g)(x^*))^-\Big)^- &= \sum_{\lambda \in \mathbf{P}_S} \operatorname{cone} \partial(\lambda g)(x^*)\\ &= \operatorname{cone}\{\varphi: \varphi \in \partial(\lambda g)(x^*) \text{ for some } \lambda \in \mathbf{P}_S\}.\end{aligned} \tag{2.10}$$

The last two expressions are closed by the compactness of the base $\mathbf{P}_S$ and the fact the $0 \notin \partial(\lambda g)(x^*)$, a compact convex set.

It only remains to show that

$$\overline{\text{cone}}(G - x^*) = \bigcap_{\lambda \in \mathbf{P}_S} \{d : (\lambda g)^0(x^*, d) \leq 0\}. \tag{2.11}$$

Containment ($\subset$) is clear. Now let d be an element of the right-hand side of (2.11). Let $e = \hat{x} - x^*$. Then

$$\alpha e + (1 - \alpha) d \in \text{cone}(G - x^*) \tag{2.12}$$

for all $0 < \alpha < 1$. Therefore, d is an element of the left-hand side of (2.11). □

Lemma 2.4. *If $A : X \to Y$ is a continuous linear operator with closed range,*

$$G = \{x : Ax - b \in -S\},$$

and $Ax^* - b \in -S$, then

(i) $\quad (G - x^*)^+ = \text{range of } A^*, \qquad (2.13)$

if $S = \{0\}$;

(ii) $\quad (G - x^*)^- = \{A^*\lambda : \lambda \in S^+, \lambda(Ax^* - b) = 0\}$,

if Y is finite dimensional and S is polyhedral.

Proof. The proof of (i) follows from the fact that the left-hand side of (2.13) is equal to the orthogonal complement of the null space of A and the range of A^* is closed when the range of A is closed. The proof of (ii) is similar to the proof of the previous lemma. □

3. The constraint qualifications

In this section we consider several constraint qualifications, including those studied in [4, 5]. We also study the notion of quasi relative interior introduced in [4] and introduce the notion of strong quasi relative interior. We first define the generalizations of relative interior.

Definition 3.1 [4]. For convex $C \subset X$, the *quasi relative interior* of C (qri C) is the set of those $x \in C$ for which $\overline{\text{cone}}(C - x)$ is a subspace.

We will use the following *constraint qualification*:

(CQ1) $\quad \exists \hat{x} \in \text{qri}\, C \quad \text{s.t.} \quad g_1(\hat{x}) \in -\text{int}\, S_1, \; g_2(\hat{x}) \in -S_2,$

and refer to it as (CQ1). If we replace qri by ri (the relative interior), then we get the usual generalized Slater condition for the abstract convex program. Note that if Y_1 is finite dimensional, then we can replace int by ri, since we could add affine constraints to restrict g_1 to lie in the span of S_1.

(CQ1) with $g=g_2$ is the constraint qualification studied in [4]. It is used along with the g affine, S polyhedral and Y finite dimensional assumptions. We present several equivalent forms of (CQ1) and its relation with the simple constraint qualification used in [5]. We now introduce a new definition which generalizes the notion of relative interior and will allow us to get parallel results for more general infinite programs. Note that we no longer take the closure of the generated cone.

Definition 3.2. For convex $C\subset X$, the *strong quasi relative interior* of C is the set of those $x\in C$ for which $\text{cone}(C-x)$ is a closed subspace.

We now get the stronger constraint qualification

$$\text{(SCQ1)}\quad \exists \hat{x}\in \text{sqri}\, C \quad \text{s.t.} \quad g_1(\hat{x})\in -\text{int}\, S_1,\ g_2(\hat{x})\in -S_2.$$

The notion of qri is studied extensively in [4]. We include several useful properties which we shall need in the sequel.

Proposition 3.1. *Let X be locally convex, $C\subset X$ be convex, and $\hat{x}\in C$.*

(a) [4, Proposition 2.4] *If X is a finite dimensional normed vector space, then* $\text{qri}\, C=\text{ri}\, C$.

(b) [4, Proposition 2.8] $\hat{x}\in \text{qri}\, C$ *if and only if* $(C-\hat{x})^+$ *is a subspace of* X^*.

(c) [4, Proposition 2.16] qri C *equals exactly the nonsupport points of* C.

(d) [4, Proposition 2.5] *Suppose* $X=\prod_{i=1}^k X_i$, *the product of topological vector spaces, and* $C_i\subset X_i$, $i=1,\dots,k$, *are convex, then* $\text{qri}\prod_{i=1}^k C_i=\prod_{i=1}^k \text{qri}\, C_i$.

(e) [4, Proposition 2.7] *Suppose* $A:X\to \mathbf{R}^n$ *is a continuous linear map. Then* $A(\text{qri}\, C)\subset \text{ri}(AC)$. □

In finite dimension, $x\in \text{ri}\, C$ if and only if $\text{cone}(C-x)$ is a subspace. Moreover, every subspace in finite dimensions is closed. Thus we see that qri and ri coincide in finite dimensions. We get a stronger result for sqri. (Note that every finite dimensional convex set is ideally convex.)

Proposition 3.2. *Let $C\subset X$ be an ideally convex set and* $\text{sqri}\, C\neq\emptyset$. *Then*

$$\text{sqri}\, C=\text{ri}\, C=\text{icr}\, C, \tag{3.1}$$

where icr *denotes the intrinsic core of C, see e.g.*, [11].

Proof. Without loss of generality, suppose that $0\in \text{sqri}\, C$. Then

$$\text{cone}(C)=\text{cone}(C-C) \text{ is a closed subspace.}$$

We can assume that this subspace is all of X and talk about core and int. But core equals sqri in this case, by definition, and it equals int for ideally convex sets, see [11]. □

The qri and sqri provide generalizations of the Slater constraint qualification to infinite dimensions. These are primal conditions using feasible points in qri C or in sqri C. The dual approach, used in [5], requires the minimal faces of C and S and their polar cones, which we now introduce.

Definition 3.3. (a) C^{f} denotes the (unique) smallest face of C containing the feasible set.

(b) For $i=1,2$, S_i^{f} denotes the (unique) smallest face of S_i containing $-g_i(F)$.

We let $\mathbb{P}_{S_i}$ be a generating set for S_i^+, $i=1,2$, i.e.,

$$\overline{\text{cone}}\, \mathbb{P}_{S_i} = S_i^+. \tag{3.2}$$

Similarly $\mathbb{P}_{C-\bar{c}}$ is a generating set for $(C-\bar{c})^+$.

Definition 3.4. Let $\bar{c} \in F$. Then

(a) $\mathbb{P}_C^{=} = \{\varphi \in \mathbb{P}_{C-\bar{c}}\colon \varphi \in (D-\bar{c})^- \backslash (C-\bar{c})^-\}$.

(b) For $i=1,2$, $\mathbb{P}_{S_i}^{=} = \{\varphi \in \mathbb{P}_{S_i}\colon \varphi \in (S_i^{\mathrm{f}})^- \backslash S_i^-\}$.

These sets differ slightly from the ones defined in [5, 6] in that we do not include points in $(C-\bar{c})^\perp$ (or $S_i^\perp$). The sets are equivalent in the case that $\overline{\text{cone}}(C-C) = X$ (or $\overline{S_i - S_i} = Y$, resp.). The sets are a generalization of the indices of the implicit equality constraints used in [2], as well as in [6, 7] and other places. Since the constraint $g(x) \in -\bar{S}$ holds if and only if $\varphi g(x) \leq 0\ \forall \varphi \in \mathbb{P}$, where $\mathbb{P}$ is a generating set for S^+, we can consider the set $\mathbb{P}$ as being the index set of constraints $g_\varphi = \varphi g$. We define the constraint qualification

(CQ2) $\quad \exists \hat{x} \in F \quad \text{s.t.} \quad g_1(\hat{x}) \in -\text{int}\, S_1$ and $\mathbb{P}_C^{=} = \emptyset$.

Proposition 3.3. *The equality set*

$$\mathbb{P}_C^{=} \subset (C-\bar{c})^+ \cap (C^{\mathrm{f}}-\bar{c})^\perp \backslash (C-\bar{c})^\perp. \tag{3.3}$$

Proof. Let $\varphi \in \mathbb{P}_C^{=}$. Then,

$$\varphi \in (C-\bar{c})^+ \subset (C^{\mathrm{f}}-\bar{c})^+ \subset (F-\bar{c})^+ \tag{3.4}$$

and

$$\varphi \in (D-\bar{c})^- \subset (F-\bar{c})^-.$$

Thus $\varphi \in (F-\bar{c})^\perp$ and $\varphi^\perp$ is a proper supporting hyperplane of $(C-\bar{c})$ which contains $(F-\bar{c})$. If $\varphi^\perp$ does not contain $(C^{\mathrm{f}}-\bar{c})$, then $(\varphi^\perp + \bar{c}) \cap C$ is a face of C which contradicts the definition of C^{f}. $\square$

The set $P_C^{=}$ consists of normals of proper supporting hyperplanes of C which contain the minimal face C^{f}. We now see that $P_C^{=}$ is independent of $\bar{c} \in F$ and so Definition 3.4 makes sense.

Proposition 3.4. *The equality set $P_C^=$ is independent of $\bar{c} \in F$.*

Proof. The result follows since $P_C^= \subset (C^f - \bar{c})^\perp$ and so, if $\varphi \in P_C^=$, then

$$\varphi(C-\bar{c}) = \varphi(C-x), \qquad \varphi(D-\bar{c}) = \varphi(D-x), \tag{3.5}$$

for all $x \in F$. □

We now define the constraint qualification

(CQ3) $\exists \hat{x} \in F$ s.t. $g_1(\hat{x}) \in -\mathrm{int}\, S_1$ and $C^f = C$.

(CQ2) and (CQ3), in the case $g = g_2$, C is a cone, and $C^\perp = \{0\}$, is used in [5].

Proposition 3.5. *Suppose that $C^f = C$. Then $P_C^= = \emptyset$.*

Proof. If $\varphi \in P_C^=$, then $\varphi \neq 0$ and, $C^f \subset \varphi^\perp$, a proper supporting hyperplane of C. □

Thus (CQ3) implies (CQ1). We now see that (CQ1) guarantees that there are no proper supporting hyperplanes of C which contain the minimal face C^f, i.e., (CQ1) implies that (CQ2) holds.

Proposition 3.6. *Suppose that* (CQ1) *holds. Then*

$$P_C^= = \emptyset. \tag{3.6}$$

Proof. If $\varphi \in P_C^=$, then $\varphi \neq 0$ and, by Proposition 3.3, $F \subset C^f \subset \varphi^\perp$ a proper supporting hyperplane of C. Thus each point of F is a support point of C, which contradicts the definition of (CQ1). □

It is clear that the set of sqri points is a subset of the set of qri points. Thus

$$\text{(SCQ1)} \;\Rightarrow\; \text{(CQ1)} \;\Rightarrow\; \text{(CQ2)} \tag{3.7}$$

i.e., the constraint qualifications guarantee that the set C does not have a proper supporting hyperplane which contains the minimal face C^f. We shall see that $\mathbb{P}_C^= = \emptyset$ is one of the conditions needed to guarantee a Lagrange multiplier for (P). A possibly weaker condition to guarantee that $\mathbb{P}_C^= = \emptyset$ is given by the following.

Proposition 3.7. *Suppose that*

$$K = \overline{\mathrm{cone}}(C-D) \text{ is a subspace.} \tag{3.8}$$

Then

$$\mathbb{P}_C^= = \emptyset.$$

Proof. First, let us show that $C^+ \cap D^- \subset K^\perp$. Let $\varphi \in C^+ \cap D^-$ and $k \in K$. Without loss of generality, $k \neq 0$. Then $k = \lim k_n$ with $k_n = \alpha_n(c_n - d_n)$, $\alpha_n \geq 0$, $c_n \in C$, $d_n \in D$. We see that

$$\varphi k_n = \alpha_n(\varphi c_n - \varphi d_n) \geq 0 \Rightarrow \varphi k \geq 0.$$

But, since K is a subspace, we get that $\varphi k = 0$.

Now, if $\varphi \in \mathbf{P}_C^=$, then $\varphi \notin C^\perp$ by Proposition 3.3, and so $\varphi \notin K^\perp$. But $\varphi \in C^+ \cap D^-$, by definition of $\mathbf{P}_C^=$, contradiction. □

The above suggests the following constraint qualification

(CQ4) $\exists \hat{x} \in F$ s.t. $g_1(\hat{x}) \in -\operatorname{int} S_1$, $\overline{\operatorname{cone}}(C - D)$ is a subspace.

and the stronger

(SCQ4) $\exists \hat{x} \in F$ s.t. $g_1(\hat{x}) \in -\operatorname{int} S_1$, $\operatorname{cone}(C - D)$ is a closed subspace.

We see that (SCQ4) and (CQ4) are related in a similar fashion to (SCQ1) and (CQ1). We will also use the following constraint qualification

(CQ5) $\exists \hat{x} \in F$ s.t. $g_1(\hat{x}) \in -\operatorname{int} S_1$, $b \in \operatorname{sqri}(A(C))$,

which generalizes the usual interior condition that $b \in \operatorname{int}(AC)$, see e.g. [13]. We will see, in Section 5, that (CQ5) is the property that connects (CQ1) for partially finite programs and the extensions to more general infinite programs. In fact, (CQ1) for partially finite programs is equivalent to (CQ5), while (CQ5) is a valid constraint qualification for more general infinite programs and extends the classical $b \in \operatorname{int}(A(C))$ constraint qualification.

4. Characterization of optimality I

We now present a duality result for extended partially finite programs (P). In actuality, we first relax the affine and finite dimensional assumptions to that of local (weak-*) compactness.

Theorem 4.1. *Suppose that* x^* *is feasible and* (CQ2) *holds for* (P), *i.e.,*

$$\exists \hat{x} \in F \ \text{ s.t. } \ g_1(\hat{x}) \in -\operatorname{int} S_1 \ \text{ and } \ \mathbf{P}_C^= = \emptyset.$$

Furthermore, suppose that the cone

$$(D - x^*)^+ \ \text{is locally compact,} \tag{4.1a}$$

and

$$(D - x^*)^- = \operatorname{cone}\{\varphi : \varphi \in \partial(\lambda g_2)(x^*),\ \lambda \in S_2^+,\ \lambda g_2(x^*) = 0\}. \tag{4.1b}$$

Then x^* *solves* (P) *if and only if the* (*Karush-Kuhn-Tucker*) *system*

$$0 \in \partial f(x^*) + \partial(\lambda_1 g_1)(x^*) + \partial(\lambda_2 g_2)(x^*) - (C - x^*)^+,$$
$$\lambda = (\lambda_1, \lambda_2) \in S^+, \quad \lambda g(x^*) = 0 \tag{4.2}$$

is consistent.

Proof. We can restate (P) as

$$\inf f(x) \quad \text{s.t.} \quad x \in C \cap D \cap E.$$

Then, x^* solves (P) if and only if

$$0 \in \partial f(x^*) - (C \cap D \cap E - x^*)^+. \tag{4.3}$$

But,

$$(C \cap D \cap E - x^*)^+ = ((C - x^*) \cap (D - x^*))^+ + (E - x^*)^+,$$

by Lemma 2.1(a) and the interiority condition in the (CQ2) assumption. Moreover,

$$((C - x^*) \cap (D - x^*))^+ = (C - x^*)^+ + (D - x^*)^+,$$

since (CQ2) implies that $((C - x^*)^- \cap (D - x^*))^+ = \{0\}$ and the second cone is locally compact, see Lemma 2.1(b). The result now follows from Lemma 2.3 and (4.1). □

We have made four assumptions to guarantee the existence of Lagrange multipliers $\lambda = (\lambda_1, \lambda_2)$ in (4.2). First, the constraint qualification (CQ2) involves two assumptions. The $\mathbf{P}_C^= = \emptyset$ condition guarantees the recession condition $C^+ \cap D^- = \{0\}$. To conclude that the sum of the polar cones is closed, we use the local compactness condition. Finally, to represent the elements of the polar cone by elements in the subdifferential, we use (4.1b). This guarantees the existence of the Lagrange multiplier λ_2 for the constraint with g_2. The interiority condition in (CQ2) guarantees the Lagrange multiplier λ_1 for the constraint g_1. This guarantees both the closure of the sum of the appropriate polar cones as well as the representation in terms of subdifferentials, see Lemma 2.3. The basic proof technique is to use the optimality condition (4.3), apply (2.5) to get a sum of cones, and then apply the constraint qualifications to guarantee that the sum is closed. This theme is followed throughout the paper. It is also the approach used in e.g., [8, 11], as well as many other papers. In [18], the so-called CHIP condition is studied which guarantees that the sum in (4.3) can be split up.

The assumption that $(D - x^*)^+$ is locally compact (along with the qri condition) guarantees closure in the sum. This is a condition in the dual space X^*. (In the next section we consider parallel results using a condition in the primal space X (along with the sqri condition).) One obvious case where the local compactness condition is satisfied is the case of partially finite programs.

Corollary 4.1. *Suppose that* (P) *is a partially finite program in Theorem* 4.1, *i.e.,* Y_2 *is finite dimensional,* S_2 *is polyhedral, and* g_2 *is affine. Then* (4.1) *holds and so,* (4.2) *characterizes optimality for feasible* x^*.

Proof. By the hypothesis, the cone$(D - x^*)^+$ is a finite dimensional polyhedral cone and so is locally compact, see Lemma 2.4. □

In the case that $g = g_2$, C is a convex cone, and $C^\perp = \{0\}$, the above corollary is given in [5]. We also recover results in [4].

Corollary 4.2 [4]. *Suppose that in Corollary* 4.1, $g = g_2$ *and* (*CQ*2) *is replaced by* (CQ1). *Then* (4.1) *holds and* (4.2) *characterizes optimality for feasible* x^*.

Proof. The proof follows from the fact that (CQ1) implies (CQ2). □

The above corollary is proved in [4] using conjugate duality. It is also presented in the case that the infimum is unattained. Our proof technique in this section shows that the finite dimensionality of Y_2 and affiness of g_2 are not essential. Rather, they can be replaced by the local compactness assumption.

5. Characterization of optimality II

In this section we establish duality results for (P) by using the notion of strong quasi relative interior introduced in Section 3. We see that (CQ5) gives the connection between the qri and sqri conditions and thus the condition needed to remove the finite dimensional assumption in the partially finite programs. We restrict g_2 to be affine and consider the infinite program

(PI) $\quad \mu = \inf\{f(x) : g_1(x) \in -S_1, g_2(x) = Ax - b = 0, x \in C\},$

where now $A : X \to Y_2$ is a continuous linear operator. We could consider the more general constraint $Ax - b \in -S_2$. This more general constraint can be simplified to the one in (PI) by adding a slack variable. Recall that $x \in \operatorname{sqri} C$ if $\operatorname{cone}(C-x)$ is a closed subspace; while $x \in \operatorname{qri} C$ if $\overline{\operatorname{cone}}(C-x)$ is a closed subspace.

Proposition 5.1. *If one of the following three conditions hold*:

(i) *there exists* $\hat{x} \in \operatorname{qri} C$ *with* $A\hat{x} = b$ *and* Y_2 *finite dimensional*;

(ii) *there exists* $\hat{x} \in \operatorname{core} C$ *with* $A\hat{x} = b$ *and range of A is closed*;

(iii) *there exists* $\hat{x} \in \operatorname{int} C$ *with* $A\hat{x} = b$ *and range of A is closed*; *then, we get*

$$b \in \operatorname{sqri}(A(C)). \tag{5.1}$$

Moreover, (5.1) *or the condition*

$$\hat{x} \in \operatorname{sqri} C \quad \text{and} \quad D \text{ is finite dimensional} \tag{5.2}$$

implies that

$$\operatorname{cone}(C-D) \text{ is a closed subspace}, \tag{5.3}$$

which, in turn, if C is closed, implies that

$$C^+ + D^+ \text{ is closed.}$$

Proof. That (i) implies (5.1) follows from Proposition 3.1(e) and Proposition 3.2. (Recall that a finite dimensional convex set is ideally convex.) To show that (ii) (or (iii)) implies (5.1) we note that

$$\begin{aligned}\operatorname{cone}(A(C))-b&=\operatorname{cone}(A(C)-A\hat{x})\\&=\operatorname{cone}(A(C-\hat{x}))\\&=A(\operatorname{cone}(C-\hat{x}))\\&=A(X)=\text{range of } A,\end{aligned}\tag{5.4}$$

which is a closed subspace.

Now, if (5.1) holds then, since D is the inverse image of b under A, we see that $\operatorname{cone}(A(C)-b)=A(\operatorname{cone}(C-D))$ is a closed subspace. Therefore, $\operatorname{cone}(C-D)$ is a closed subspace by continuity of A. If (5.2) holds, then $\operatorname{cone}(C-D)=(\operatorname{cone}(C-\hat{x})-(D-\hat{x}))$ is the sum of a closed subspace and a finite dimensional subspace and so is closed, by Lemma 2.2(c). That $C^{+}+D^{+}$ is closed follows from Lemma 2.2(a). □

We now see that one of the constraint qualifications (CQ5) or (SCQ4) and the standard closed range assumption on A are sufficient to guarantee the existence of a Lagrange multiplier for (PI). In contrast to Theorem 4.1, we use Lemma 2.2(a) to prove the result, i.e., we use a condition in the primal space X to guarantee the closure of the sum of the cones.

Theorem 5.1. *Suppose that, for* (PI), *we have one of the constraint qualifications* (CQ5) *or* (SCQ4). *In addition, C is closed and the range of A is closed. Then x^* (feasible) is optimal if and only if the system*

$$\begin{gathered}0\in\partial f(x^*)+\partial(\lambda_1 g)(x^*)-A^*\lambda_2-(C-x^*)^{+},\\ \lambda_1\in S_1^{+},\quad \lambda_1 g(x^*)=0,\quad \lambda_2\in Y^*\end{gathered}\tag{5.5}$$

is consistent.

Proof. The proof is similar to that of Theorem 4.1. We have that x^* is optimal if and only if

$$0\in\partial f(x^*)-(C\cap D\cap E-x^*)^{+}.$$

Now (SCQ4) or, by Proposition 5.1, (CQ5) implies that (5.3) holds. We can now apply Lemmas 2.1(a), 2.2(a), 2.3 and 2.4 to get (5.5). □

We can also get parallel results to the partially finite case in the previous section, i.e., we replace qri by sqri and replace the finite dimensional range space assumption with a finite dimensional assumption in the domain space.

Corollary 5.1. *Suppose that in Theorem 5.1, D is finite dimensional and the two constraint qualifications are replaced by* (SCQ1). *Then the conclusion of the theorem holds.*

Proof. The closure in Lemma 2.2(a) is guaranteed by (5.2) in Proposition 5.1. □

We now show how certain results for infinite constrained interpolation problems, studied recently in [8], can be easily derived with a weaker constraint qualification from the main result of this section. Following [8], we let X be a Hilbert space and $\{x_1, x_2, \ldots\}$ be an unconditional basis, i.e., each element x in $H = \overline{\text{span}}\{x_1, x_2\}$ has the unique representation in terms of the basis, $x = \sum_i \alpha_i x_i$ and the convergence is unconditional. Let $A: X \to \ell_2$ be defined by

$$Ax = (\langle x, x_1\rangle, \langle x, x_2\rangle, \ldots), \quad x \in X. \tag{5.6}$$

Then $A^*: \ell_2 \to X$ is given by

$$A^*(\alpha_1, \alpha_2, \ldots) = \sum_i \alpha_i x_i.$$

Consider the infinite constrained interpolation problem

(IP) $\quad \inf\{\|k - x\|: x \in C, \langle x, x_i\rangle = d_i, i = 1, 2, \ldots\},$

where $k \in X$, $C \subset X$ is a closed convex cone and $d = (d_i) \in A(C) \subset \ell_2$. Then (IP) is a problem of type (PI) with $b = d$ and $f(x) = \|k - x\|$. We define D as before, i.e., it is the inverse image of d under A.

Example 5.1. *For* (IP), *suppose that* cone$(C - D)$ *is a closed subspace. Then,* (IP) *attains its infimum at some point $x_0 \in C$, and*

$$\|k - x_0\| = \min\{\|k + A^*\lambda - x_0\|: k \in C\} \tag{5.7}$$

for some $\lambda \in \ell_2$.

Proof. Since the objective function of (IP) is coercive, i.e., its value converges to infinity as $\|x\| \to \infty$, the infimum is attained, see e.g. [12]. Also, the fact that the range of A^* is closed is shown in [8]. (This implies that the range of A is also closed.) The result follows from the theorem if we note that the optimality conditions for the right-hand side of (5.7), after squaring the objective function, are the same as for the original problem, after squaring the norm. □

The above result also holds if the subspace H has finite dimensional codimension, i.e., if the operator A has a finite dimensional null space. This follows from Corollary 5.1.

6. Conclusion

We have studied the convex program (P) defined in Section 2. Under suitable assumptions, we have provided various constraint qualifications for (P). This resulted in two types of multiplier theorems to characterize optimality of a feasible point x^*. (Note that g_2 is not necessarily affine.)

$$\text{(LM1)}\quad \begin{aligned} &0\in\partial f(x^*)+\partial(\lambda_1 g_1)(x^*)+\partial(\lambda_2 g_2)(x^*)-(C-x^*)^+,\\ &\lambda=(\lambda_1,\lambda_2)\in S^+,\quad \lambda g(x^*)=0\end{aligned} \tag{6.1}$$

is consistent, and

$$\text{(LM2)}\quad \begin{aligned} &0\in\partial f(x^*)+\partial(\lambda_1 g_1)(x^*)+A^*\lambda_2-(C-x^*)^+,\\ &\lambda=(\lambda_1,\lambda_2)\in S^+,\quad \lambda_1 g_1(x^*)=0,\quad S_2=0\end{aligned} \tag{6.2}$$

is consistent.

We summarize the constraint qualifications here for convenience:

(CQ) $\exists\hat{x}\in F\cap\operatorname{int} C$ with $g(\hat{x})\in-\operatorname{int} S$ (Slater's condition),

(CQ1) $\exists\hat{x}\in F\cap\operatorname{qri} C$ with $g_1(\hat{x})\in-\operatorname{int} S_1$,

(SCQ1) $\exists\hat{x}\in F\cap\operatorname{sqri} C$ with $g_1(\hat{x})\in-\operatorname{int} S_1$,

(CQ2) $\exists\hat{x}\in F$ with $g_1(\hat{x})\in-\operatorname{int} S_1$ and $\mathbf{P}_C^=\emptyset$,

(CQ3) $\exists\hat{x}\in F$ with $g_1(\hat{x})\in-\operatorname{int} S_1$ and $C^f=C$,

(CQ4) $\exists\hat{x}\in F$ with $g_1(\hat{x})\in-\operatorname{int} S_1$ and $\overline{\operatorname{cone}}(C-D)$ is a subspace,

(SCQ4) $\exists\hat{x}\in F$ with $g_1(\hat{x})\in-\operatorname{int} S_1$ and $\operatorname{cone}(C-D)$ is a closed subspace,

(CQ5) $\exists\hat{x}\in F$ with $g_1(\hat{x})\in-\operatorname{int} S_1$ and $b\in\operatorname{sqri}(A(C))$.

These constraint qualifications needed additional assumptions taken from the following:

(AS1) $g_2(x)=Ax-b$ (affine constraint),

(AS2) A has closed range,

(AS3) S_2 is polyhedral,

(AS4) Y_2 is finite dimensional,

(AS5) C is closed,

(AS6) $(D-x^*)^+$ is locally compact,

(AS7) the subdifferentiability representation (4.1b) holds,

(AS8) $S_2=\{0\}$,

(AS9) D is a finite dimensional affine space.

We now summarize the various constraint qualifications and the accompanying assumptions in Table 1.

Table 1

Constraint qualifications and accompanying assumptions

	(CQ1)	(SCQ1)	(CQ2)	(CQ3)	(CQ4)	(SCQ4)	(CQ5)
(LM1)	(AS6) (AS7)	× ×	(AS6) (AS7)	(AS6) (AS7)	(AS6) (AS7)	× ×	× ×
(LM2)	(AS1) (AS3) (AS4)	(AS1) (AS2) (AS5) (AS8) (AS9)	(AS1) (AS3) (AS4)	(AS1) (AS3) (AS4)	(AS1) (AS3) (AS4)	(AS1) (AS2) (AS5) (AS8)	(AS1) (AS2) (AS5) (AS8)

Applications of partially finite programs to constrained approximation as well as to many other areas are given in [4]. The above allows extensions to nonlinear interpolation as well as infinite dimensional interpolation problems. In particular, we get parallel results to the partially finite case if we use sqri and D finite dimensional. For example, this case arises if A is a Fredholm operator of the second kind, since these operators have finite dimensional null space. We could also treat nonlinear constrained interpolation problems rather than just linear ones, see e.g. [14]. If the constraints are differentiable, then we could get multiplier rules by linearization and applying the above results. We could then use sequential quadratic programming (SQP) in infinite dimensional spaces. The individual steps in the SQP methods will be tractable as they will fall into the partially finite class of problems. These numerical results will be studied in a future paper.

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