A Strengthened Barvinok-Pataki Bound on **SDP** Rank

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Abstract

The Barvinok-Pataki bound provides an upper bound on the rank of extreme points of a spectrahedron. This bound depends solely on the number of affine constraints of the problem, i.e., on the algebra of the problem. Specifically, the triangular number of the rank r is upper bounded by the number of affine constraints. We revisit this bound and provide a strengthened upper bound on the rank using the singularity degree of the spectrahedron. Thus we bring in the geometry and stability of the spectrahedron, i.e., increased instability as seen by higher singularity degree, yields a lower, strengthened rank bound.

Keywords: Barvinok-Pataki bound, facial reduction, rank, singularity degree, Semidefinite programming.

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Contents

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1 Introduction

The Barvinok-Pataki bound [1,7] shows that an extreme point X of a spectrahedron with rank r satisfies $t(r) \leq m$, where $t(r)$ is the *triangular number*, and m is the number of affine constraints. Thus we get that extreme points have a bound on the rank that depends *solely* on the number of constraints, the algebra of the linear manifold that define the spectrahedron. We revisit this bound and provide a strengthened bound on the rank by adding information from the *singularity degree* of the spectrahedron, i.e., the minimum number of facial reduction steps to obtain strict feasibility; see e.g., the survey [5]. Thus we see that this new bound depends not only on the number of affine constraints but also on the *geometry and stability* of the spectrahedron; see Theorem 2.11.

1.1 Background and Notation

We let: $\mathbb{R}^n, \mathbb{R}^{m\times n}$ be the standard real spaces of n-vectors and m-by-n matrices, respectively; $\mathbb{S}^n$ denotes the Euclidean space of $n \times n$ symmetric matrices equipped with the trace inner-product; the cone of positive semidefinite (definite) matrices is denoted by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$), respectively, and we use the standard partial order notation $X \succeq 0$ ($X \succ 0$), respectively. We use relint to denote the relative interior.

For $A_i \in \mathbb{S}^n,\ i \in \{1,\dots,m\}$, we define the linear map $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$, $\mathcal{A}(X) = (\langle A_i, X\rangle)_i \in \mathbb{R}^m$. A *spectrahedron,* $\mathcal{F}$ is the intersection of an affine set and the positive semidefinite cone. Given $b \in \mathbb{R}^m$ and a linear map $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$, we represent the spectrahedron as

$$\mathcal{F} = \{X \succeq 0 : \mathcal{A}(X) = b\}.$$

Without loss of generality, we assume that $\mathcal{A}$ is onto (surjective). We use $\mathcal{F}$ to denote a spectrahedron throughout this manuscript. Spectrahedra are commonly used for constraint systems in *semidefinite programming (**SDP**)*. Given a convex function f from $\mathbb{S}^n$ to $\mathbb{R}$, a semidefinite program is an optimization problem over a spectrahedron:

$$\begin{array}{rcl} p^* = & \min\limits_{X\in\mathbb{S}^n} & f(X) \\ & \text{s.t.} & \mathcal{A}(X) = b \in \mathbb{R}^m \\ & & X \succeq 0. \end{array} \tag{1.1}$$

Studying the rank, and in particular obtaining low rank solutions, is important in many applications. For example, for **SDP** relaxations of protein folding problems, rank three solutions are important because molecules sit in the three-dimensional space. The Barvinok-Pataki bound provides a target for finding a low rank solution of a feasible point of a spectrahedron. Note that for a linear **SDP**, $f(X) = \operatorname{trace} CX$, we can add the constraint $f(X) = p^*$ to apply the rank bound for optimal solutions. Another application is the highly cited paper [3] that exploits knowledge about a low rank optimal solution to reduce the dimensions using the substitution $X = VV^T$, where $V \in \mathbb{R}^{n\times r}$.

2 A Strengthened Barvinok-Pataki Bound

In this section we present an improved Barvinok-Pataki bound, see Theorem 2.11. We work with a nonempty spectrahedron. The *triangular number, $t(n)$*, is defined as $t(n) = \binom{n+1}{2} = n(n+1)/2$. Note that the surjectivity of $\mathcal{A}$ implies that the triangular number $t(n) \geq m$.

2.1 Known Bounds

We recall the following definitions and results. A convex cone F is a *face* of $\mathbb{S}^n_+$ if

$$\text{for } X, Y \in \mathbb{S}^n_+ \text{ with } \{\lambda X + (1-\lambda)Y : \lambda \in (0,1)\} \subseteq F, \text{ we have } X, Y \in F.$$

Theorem 2.1 ([7, Theorem 2.1])**.** *Suppose that $X \in F$, where F is a face of the feasible set of (1.1). Let $d = \dim F$, $r = \operatorname{rank} X$. Then*

$$t(r) \leq m + d. \tag{2.1}$$

Theorem 2.2 ([1, Theorem 1.1])**.** *Let $\mathcal{L} \subset \mathbb{S}^n$ be an affine manifold such that the intersection $\mathcal{F} = \mathbb{S}^n_+ \cap \mathcal{L} \neq \emptyset$ and $\operatorname{codim} \mathcal{L} \leq t(r+1) - 1$ for some nonnegative integer r. Then there exists $X \in \mathcal{F}$ such that $\operatorname{rank} X \leq r$.*

Theorem 2.3 ([1, Theorem 1.2])**.** *Let $r > 0, n \geq r+2$. Let $\mathcal{L} \subset \mathbb{S}^n$ be an affine manifold such that the intersection $\mathcal{F} = \mathbb{S}^n_+ \cap \mathcal{L} \neq \emptyset$ and bounded, and $\operatorname{codim} \mathcal{L} = t(r+1)$, for some nonnegative integer r. Then there exists $X \in \mathcal{F}$ such that $\operatorname{rank} X \leq r$.*

Remark 2.4. *Theorems 2.1 to 2.3 all concern bounds on the rank of a feasible point to a spectrahedron. We continue with some remarks for the three theorems above.*

Given the number of constraints, Theorem 2.1 gives an upper bound on the rank of a solution. The most well-known application of Theorem 2.1 is the case of extreme points. An extreme point X of a convex set C is a point that cannot be expressed as a convex combination of any two distinct points in C. The minimal face containing an extreme point X is 0-dimensional, i.e., $\dim(\operatorname{face}(\{X\})) = 0$. From (2.1), we conclude that

$$t(\operatorname{rank}(X)) \leq m, \quad \textit{for all extreme points } X \in \mathcal{F}. \tag{2.2}$$

Theorem 2.2 is a consequence of [2, Theorem 1.3] which can be interpreted as follows. For the feasible constraint system of (1.1), there is a solution X such that its rank is bounded by $\lfloor \frac{\sqrt{8m+1}-1}{2} \rfloor$. We may obtain an equivalent bound by defining the smallest $r \in \mathbb{N}$ satisfying $\binom{r+2}{2} > m$. Therefore if we have $\binom{r+2}{2} - 1 \geq m$, where m is the number of linearly independent constraints, we obtain the statement in Theorem 2.2.

Theorem 2.3 is stated with a bounded spectrahedron. Suppose that we are given a triple (r, m, n), where r is an upper bound on the target rank; $m = \binom{r+2}{2}$ is the number of linearly independent constraints; and the embedding space $\mathbb{S}^n$ such that $n \geq r+2 \geq 3$. Then there exists a point $X \in \mathcal{F}$ such that $\operatorname{rank}(X) \leq r$.

In this note the *Barvinok-Pataki bound* refers to (2.2).

Theorem 2.5. *(Barvinok-Pataki bound [1, 7]) Every extreme point $X \in \mathcal{F}$ satisfies $t(\operatorname{rank}(X)) \leq m$.*

2.2 Facial Reduction

The *minimal face* of $C \subseteq \mathbb{S}^n_+$, $\operatorname{face}(C)$, is the intersection of all faces containing C. A face F is *exposed* if it is the intersection of $\mathbb{S}^n_+$ and a hyperplane. In other words, F admits the representation $F = \mathbb{S}^n_+ \cap Z^\perp$, for some $Z \in \mathbb{S}^n_+$. The vector Z is called an *exposing vector* of F. An exposing vector is *maximal* if it is of the highest rank over all exposing vectors.

Proposition 2.6. *(theorem of the alternative) For the feasible constraint system of (1.1), exactly one of the following statements holds:*

1. *There exists $X \succ 0$ such that $\mathcal{A}(X) = b$,*

2. *There exists $y \in \mathbb{R}^m$ such that*

$$\mathcal{A}^*(y) \in \mathbb{S}^n_+ \setminus \{0\}, \ \langle b, y\rangle = 0. \tag{2.3}$$

*Facial reduction (**FR**)* is a process of identifying the minimal face of $\mathbb{S}^n_+$ containing the affine set $\{X : \mathcal{A}(X) = b\}$. Since $\mathbb{S}^n_+$ is facially exposed, the process can be characterized as identifying an exposing vector. Algorithm 2.1 below is a pseudo code for **FR** algorithm. More details can be found in [5,8,10]. Finding an exposing vector Z in Algorithm 2.1 is generally done by solving the

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Algorithm 2.1 Pseudo Code for Facial Reduction Algorithm

Require: data $(\mathcal{A}, b)$ for affine set $\{X : \mathcal{A}(X) = b\}$
  while $\nexists X \succ 0$ satisfying $\mathcal{A}(X) = b$ **do**
    find an exposing vector Z
    compute V such that $\text{range}(V) = \text{null}(Z)$
    $\mathcal{A} \leftarrow \mathcal{A}_V(\cdot) := \mathcal{A}(V(\cdot)V^T)$
  end while

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auxiliary problem (2.3). After **FR** we obtain the minimal face of $\mathbb{S}^n_+$ containing $\{X : \mathcal{A}(X) = b\}$ and this minimal face has the representation $V\mathbb{S}^r_+V^T$ for some $r \leq n$. We call this vector V a *facial vector*. A *minimal facial vector* is the one with the minimum number of columns.

It is known that every **FR** step results in at least one constraint becoming redundant, see e.g., [10, Section 3.5]. We give a short proof below.

Lemma 2.7. *At least one linear constraint of the **SDP** becomes redundant after each step of **FR**.*

Proof. Let $\mathcal{A}^*(y)$ be the exposing vector satisfying the system (2.3). Let V be a minimal facial vector satisfying $\text{null}(\mathcal{A}^*(y)) = \text{range}(V)$. Clearly, $V^T\mathcal{A}^*(y)V = \sum_{i=1}^m y_i V^T A_i V = 0$. After the reduction the constraints have the form $\text{trace}(V^T A_i V X) = b_i, \ \forall i$. Since $y \in \mathbb{R}^m$ is a nonzero vector, the matrices in $\{V^T A_i V\}_{i=1,\dots,m}$ are not linearly independent. □

By Lemma 2.7 we may remove redundant constraints in each **FR** step and proceed to the next iteration. Lemma 2.7 plays an important role in obtaining the tighter bound in Section 2.3. For the minimum length of **FR** iterations, we give a special name.

Definition 2.8. *[9,11] Given a spectrahedron $\mathcal{F}$, the singularity degree of $\mathcal{F}$, denoted by $\text{sd}(\mathcal{F})$, is the smallest number of facial reduction steps for finding $\text{face}(\mathcal{F})$.*

It is known that if we find an exposing vector Z in $\text{relint}(\{X \succeq 0 : \mathcal{A}(X) = b\})$ at every iteration, the number of **FR** steps is equal to $\text{sd}(\mathcal{F})$. The singularity degree has an obvious upper bound n. In fact the singularity degree admits a tighter upper bound.

Lemma 2.9. *[9,10] Let $\mathcal{F}$ be a nonempty spectrahedron such that $\mathcal{F} \neq \{0\}$. Then the singularity degree of $\mathcal{F}$ satisfies the following bound:*

$$\text{sd}(\mathcal{F}) \leq \min\{n-1, m\}.$$

We close this section by showing that the rank of feasible points are unchanged after facial reduction.

Lemma 2.10. *Let $V \in \mathbb{R}^{n\times r}$ be a minimal facial vector containing the set $\mathcal{F} := \{X \succeq 0 : \mathcal{A}(X) = b\}$, i.e., $V\mathbb{S}^r_+V^T \supseteq \mathcal{F}$. Then, for VRV^T feasible, we have* $\operatorname{rank}(VRV^T) = \operatorname{rank}(R)$.

Proof. Suppose that $\operatorname{rank}(R) = r$. Then R has the spectral decomposition $R = \sum_{i=1}^r \lambda_i x_i x_i^T$ and

$$VRV^T = \sum_{i=1}^r \lambda_i V x_i (V x_i)^T.$$

Let $X_r = \begin{bmatrix} x_1 & \cdots & x_r \end{bmatrix}$ and consider the equation $\begin{bmatrix} Vx_1 & \cdots & Vx_r \end{bmatrix} a = VX_r a = 0$ for $a \in \mathbb{R}^r$. Then we have

$$V^T V X_r a = 0 \implies X_r a = 0 \implies X_r^T X_r a = 0 \implies a = 0.$$

Thus VRV^T is the sum of rank one matrices that are linearly independent and hence $\operatorname{rank}(VRV^T) = \operatorname{rank}(R)$. ☐

2.3 The Improved Bound

In this section we present the strengthened Barvinok-Pataki bound. By Lemma 2.10, the ranks of feasible points of the original spectrahedron are completely determined by the ranks of feasible points in the facially reduced spectrahedron. Using Lemma 2.7 and Lemma 2.9 we obtain the main result of this manuscript; a tighter upper bound on rank by including the singularity degree.

Theorem 2.11. *(A strengthened Barvinok-Pataki bound) Suppose that the singularity degree of the nonempty spectrahedron $\mathcal{F}$ satisfies $s = \operatorname{sd}(\mathcal{F}) > 0$. Then there exists a point $X \in \mathcal{F}$ with $r = \operatorname{rank}(X)$ that satisfies*

$$t(r) \leq \min\{t(n-s), m-s\}. \tag{2.4}$$

Proof. From Lemma 2.7, we have at most $m-s$ linearly independent constraints after **FR**. Then the upper bound $m-s$ follows from Theorem 2.5. Since each **FR** step reduces the variable size by at least 1, we have $r \leq n-s$. Since t is monotonic on the positive real line, $t(r) \leq t(n-s)$ follows.

We now show (2.4) is well-defined. Suppose that $\mathcal{F} = \{0\}$. Then the inequality clearly holds. We may assume that $\mathcal{F} \neq \{0\}$. Due to Lemma 2.9, $t(n-s)$ is positive. Now suppose that $m \leq n-1$. Then $\operatorname{sd}(\mathcal{F}) \leq m$. Suppose to the contrary $\operatorname{sd}(\mathcal{F}) = m$. We note that Lemma 2.7 leads to $\mathcal{F} = \mathbb{S}^n_+$, which contradicts the assumption. Thus, we obtain $\operatorname{sd}(\mathcal{F}) \leq m-1$ and $m-s$ is positive in this case. Now suppose that $m > n-1$. Then $m-s > n-s-1 \geq 0$, where the last inequality follows from Lemma 2.9. ☐

Remark 2.12. *From the proof of Theorem 2.11, we note that we may obtain a tighter bound on the singularity degree than the one given in Lemma 2.9. If $m \leq n-1$, $\operatorname{sd}(\mathcal{F}) \leq m-1$ when $\operatorname{sd}(\mathcal{F}) > 0$.*

We now give an elementary analysis on the bound (2.4). Corollary 2.13 below gives an explicit upper bound on the rank of a solution. It follows from the definition of the triangular number and the integrality of the rank function.

Corollary 2.13. *Let $s = \operatorname{sd}(\mathcal{F})$. Then there exists a solution X to (1.1) such that*

$$\operatorname{rank}(X) \leq \left\lfloor \frac{\sqrt{1+8\min\{t(n-s)\ ,\ m-s\}}}{2} - 1 \right\rfloor.$$

Remark 2.14 below discusses the upper bound $\min\{t(n-s), m-s\}$. The minimizer of $\min\{t(n-s), m-s\}$ is determined by the relation among m, n and s.

Remark 2.14. *Let n, m, s be given. Then the following hold.*

1. *Suppose that $m - n + \frac{1}{8} \geq 0$ holds. Then*

$$\min\{t(n-s), m-s\} = \begin{cases} t(n-s) & \text{if } \left|s - \left(n - \frac{1}{2}\right)\right| \leq \frac{1}{2}\sqrt{8m - 8n + 1}, \\ m - s & \text{otherwise.} \end{cases}$$

2. *Suppose that $m - n + \frac{1}{8} \leq 0$ holds. Then $\min\{t(n-s), m-s\} = m - s$.*

Proof. Suppose that $m - n + \frac{1}{8} \geq 0$ holds. Then

$$t(n-s) \geq m - s \iff s^2 + (1-2n)s + n^2 + n - 2m \geq 0$$

Using the root formula for the quadratic function with respect to s, we note that the inequality holds only when

$$s \geq n - \frac{1}{2} + \frac{1}{2}\sqrt{8m - 8n + 1} \quad \text{or} \quad s \leq n - \frac{1}{2} - \frac{1}{2}\sqrt{8m - 8n + 1}.$$

We now suppose that $m - n + \frac{1}{8} \leq 0$ holds. Then $m - s \leq n - s - \frac{1}{8}$. We compare the numbers $n - s - \frac{1}{8}$ and $t(n-s)$:

$$t(n-s) - \left(n - s - \frac{1}{8}\right) = \frac{1}{2}\left(s^2 - 2ns + s + n^2 - n + \frac{1}{4}\right) = \frac{1}{2}\left(s - \left(n - \frac{1}{2}\right)\right)^2 \geq 0.$$

Thus, we have that

$$t(n-s) \geq n - s - \frac{1}{8} \geq m - s.$$

□

We give an elementary example to illustrate the advantage of the new improved Barvinok-Pataki bound.

Example 2.15. *Consider the spectrahedron $\mathcal{F} = \{X \in \mathbb{S}^4_+ : \mathrm{trace}(A_i X) = b_i,\ i = 1, 2, 3\}$ with the data*

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \ b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 2.5 gives two possibilities for the rank r of a solution to $\mathcal{F}$:

$$t(1) = 1 \leq 3 \quad \textit{or} \quad t(2) = 3 \leq 3.$$

We obtain an exposing vector by solving the auxiliary system (2.3) for $y \in \mathbb{R}^3$:

$$\mathcal{A}^*(y) = \begin{bmatrix} y_1 & 0 & 0 & y_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & y_3 & 0 \\ y_3 & 0 & 0 & y_2 \end{bmatrix} \in \mathbb{S}^4_+ \setminus \{0\}, \ b^T y = y_1 = 0.$$

It is easy to see that $\mathrm{Diag}([0;0;0;1])$ is a maximal exposing vector and we complete the first round *of* ***FR*** *with the minimal facial vector*

$$V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

The second constraint becomes redundant. We now have the new data

$$V_1^T A_1 V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ V_1^T A_3 V_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and proceed to the next iteration for ***FR****. By solving the auxiliary system* (2.3) *we obtain*

$$V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and } V_2^T V_1^T A_1 V_1 V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.6}$$

and the third constraint becomes redundant. We note that $\bar{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *is a Slater point and* ***FR*** *algorithm terminates. The* ***FR*** *algorithm terminated with two iterations, i.e.,* $\mathrm{sd}(\mathcal{F}) = 2$*.*

We apply Theorem 2.11 to $\mathcal{F}$*. Since* $t(n - \mathrm{sd}(\mathcal{F})) = t(4-2) = 3$ *and* $m - \mathrm{sd}(\mathcal{F}) = 3 - 2 = 1$*, we conclude that every extreme point* X *of* $\mathcal{F}$ *satisfies* $t(\mathrm{rank}(X)) \leq 1$*. The only rank satisfying this bound is* $\mathrm{rank}(X) = 1$*. The point* $\mathrm{Diag}([1;0;0;0])$ *certifies the existence of a rank* 1 *solution.*

Rank of optimal solutions contains important information in many instances of **SDP** relaxations for combinatorial optimization problems. Interesting instances include the **SDP** relaxations for the quadratic assignment problem [6] or the protein side-chain positioning problems [4]. In these problems rank 1 optimal solutions of the **SDP** relaxations give global optimal solutions for the underlying nonconvex combinatorial problems. Example 2.16 below illustrates that Theorem 2.11 provides useful information on the optimal solution rank. The upper bound we obtain in Theorem 2.11 can function as a target rank when we seek for a low rank optimal solution.

Example 2.16. *With* A_1, A_2, A_3 *defined in Example 2.15, we define additional data matrices*

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Consider the following ***SDP***

$$\begin{array}{rcl} p^* & = & \min \quad \mathrm{trace}(CX) \\ & & \quad s.t. \quad \mathrm{trace}(A_i X) = b_i, \ i = 1, \ldots, 5 \\ & & \qquad\quad X \in \mathbb{S}^4_+. \end{array} \tag{2.7}$$

In order to obtain the singularity degree of the feasible region $\mathcal{F}$ *to* (2.7) *we consider the auxiliary* *system* (2.3)

$$\mathcal{A}^*(y) = \begin{bmatrix} y_1 & y_5 & 0 & y_3 \\ y_5 & 0 & y_4 & 0 \\ 0 & y_4 & y_3 & 0 \\ y_3 & 0 & 0 & y_2 \end{bmatrix} \in \mathbb{S}^4_+ \setminus \{0\},\ b^T y = y_1 = 0.$$

It is easy to see that $\mathrm{Diag}([0;0;0;1])$ *is a maximal exposing vector and we complete the first round of* ***FR*** *with the minimal facial vector* V_1 *defined in* (2.5)*. The second constraint becomes redundant and we proceed to the next* ***FR*** *step with*

$$V_1^T A_1 V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ V_1^T A_3 V_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ V_1^T A_4 V_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},\ V_1^T A_5 V_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then a maximal exposing vector to the facially reduced spectrahedron may be chosen with $\mathrm{Diag}([0;0;1])$ *and hence we obtain the minimal facial vector* V_2 *from* (2.6)*. The third and the fourth constraints become redundant and we are left with*

$$V_2^T V_1^T A_1 V_1 V_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ V_2^T V_1^T A_5 V_1 V_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is clear that $\bar{X} = I_2$ *is feasible and positive definite. Thus we again have* $\mathrm{sd}(\mathcal{F}) = 2$.

After ***FR****, we obtain the following facially reduced* ***SDP***

$$\begin{array}{rcl} p^* & = & \min \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bullet X \\ & & \text{s.t.} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bullet X = 1,\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet X = 0 \\ & & \qquad X \in \mathbb{S}^2_+. \end{array} \tag{2.8}$$

The optimal value p^* *to* (2.8) *(and* (2.7)*) is* 0*. We now consider the singularity degree of the optimal set*

$$\mathcal{F}^* := \mathcal{F} \cap \{X : \langle C, X \rangle = 0\}.$$

By a similar approach we obtain that $\mathrm{sd}(\mathcal{F}^*) = 3$*. We note that* $t(n - \mathrm{sd}(\mathcal{F}^*)) = t(4-3) = 1$ *and* $(m+1) - \mathrm{sd}(\mathcal{F}^*) = 3$*. Thus the extreme points* X^* *of the optimal set* $\mathcal{F}^*$ *hold* $\mathrm{rank}(X^*) \leq 1$*. Therefore all extreme points of the optimal set are rank* 1*. The point* $X^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *meets the bound.*

3 Conclusion

Suppose that we are given a spectrahedron $\mathcal{F} = \{X \succeq 0 : \mathcal{A}(X) = b\}$ with $b \in \mathbb{R}^m$. The Barvinok-Pataki bound guarantees the existence of a point X satisfying $\mathrm{rank}(X) = r$ and $t(r) \leq m$. An obvious question is: when is the bound tight? We present a strengthened bound using the singularity degree $\mathrm{sd}(\mathcal{F})$

$$t(r) \leq \min \{t(n - \mathrm{sd}(\mathcal{F})),\ m - \mathrm{sd}(\mathcal{F})\} \leq m.$$

The knowledge of the strengthened bound can help obtain low rank solutions in many applications. For example, the strengthened bound can be used for reducing the variable dimensions in nonlinear methods for solving **SDP**s, e.g., [3]. We may perform low rank projections on the semidefinite cone that arise in alternating direction method of multipliers (ADMM). For **SDP** relaxations for combinatorial optimization problems, we can recover a global optimal solution for its underlying combinatorial problem if the bound on the rank is 1.

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