MATH 135, W20, Sections 001/002, Zack Cramer

**THIS WEEK AT A GLANCE**

**Week 2: Lectures 5-8.** §3.1-3.4

**Goals:**

1. Prove/disprove statements of the form $\forall x \in S, P(x) \quad \exists x \in S, P(x)$
2. Prove/disprove implications
3. State and prove some new results on integer divisibility.

**Proving $\forall x \in S, P(x)$**

→ Let $x$ be an arbitrary element of $S$.

→ Verify $P(x)$

→ Since $x$ was arbitrary, $P(x)$ holds for all $x \in S$. 
Never assume what you are trying to prove, and use lots of English words. Your proof should read like a story.

It is sometimes helpful to split your proof into cases (e.g. \( \forall x \in \mathbb{R}, |x-3|+2|x+2| \geq 5 \)). Be sure your cases cover all possible values in the domain!

**Proving \( \exists x \in S, P(x) \)**

To prove \( \exists x \in S, P(x) \), you only need to exhibit a single \( x \) in the domain satisfying \( P(x) \). If you think you’ve found such an \( x \), ALWAYS verify that it does indeed satisfy \( P(x) \). Watch out for extraneous solutions!

(e.g., \( \exists k \in \mathbb{Z}, \ln(19k+5) = \ln(k^2-15) \).)
For statements with multiple quantifiers, unpack the statement from left to right and prove accordingly.

(e.g. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x^3 - y^3 = 1$)

Disproving $\forall x \in S, P(x)$ or $\exists x \in S, P(x)$

To disprove $\forall x \in S, P(x)$, prove its negation $\exists x \in S, \neg P(x)$

To disprove $\exists x \in S, P(x)$, prove its negation $\forall x \in S, \neg P(x)$

Proving Implications

To prove an implication $A \Rightarrow B$ directly,

- Assume the hypothesis is true
- Deduce the conclusion
Divisibility of Integers

**Definition:** An integer \( n \) is said to be

(a) **even** if there exists an integer \( K \) such that \( n = 2K \).

(b) **odd** if there exists an integer \( M \) such that \( n = 2M + 1 \).

More generally...

**Definition:** Given integers \( M \) and \( n \), we say that \( M \) **divides** \( n \) (and write \( M \mid n \)) if there is an integer \( K \) such that \( n = MK \).

**Proposition (Transitivity of Divisibility (TD))**: For all integers \( a, b, c \), if \( a \mid b \) and \( b \mid c \), then \( a \mid c \).

**Proposition (Divisibility of Integer Combinations (DIC))**: For all integers \( a, b, c \), if \( a \mid b \) and \( a \mid c \), then for all integers...
\[ x \text{ and } y, \ a \mid (bx + cy) \]

(e.g. since \( 5 \mid 10 \) and \( 5 \mid 15 \), DIC states that for all integers \( x \) and \( y \), \( 5 \mid (10x + 15y) \).)

**Note:** The converse to DIC is also true! Its proof is cool because the hypothesis is of the form \( \forall x \in S, P(x) \). In this case we can pick specific values from the domain to help us deduce the conclusion.