§9.1 - Numerical Integration

Some functions out in the wild have no antiderivative

*Ex*: \( f(x) = e^{x^2} \) has no antiderivative... WEIRD!

This means that FTC cannot be used to compute \( \int_{a}^{b} e^{x^2} \, dx \), so we need to use estimates

1. Left endpoints
2. Right endpoints
3. Midpoints
4. Trapezoids

If we're going to use estimates, we should ask the following 2 questions:

1. How good is the estimate?
2. How many points are required to get a good estimate?

How Good is an Estimate?

Analysis: Left endpoint method.

The biggest factors affecting our estimate are

1. Number of rectangles used \((n)\), and
2. The shape of \( f(x) \).
   (Flatter function, better estimate)
Let's look at just 1 rectangle to start:

Let $K = \max |f'(x)|$ for $x \in [a, b]$, so $f(x)$ has slope at most $K$.

By using a line with slope $K$, we can get an easy upper bound on the error:

Error $\leq$ Area of triangle $= \frac{1}{2} \cdot \text{base} \cdot \text{height}$

$= \frac{1}{2} \cdot \Delta x \cdot (K \cdot \Delta x) = \frac{k(\Delta x)^2}{2}$
By adding up over all \( n \) rectangles, we get

\[
\text{Error} \leq \frac{K(\Delta x)^2}{2} + \frac{K(\Delta x)^2}{2} + \ldots + \frac{K(\Delta x)^2}{2} = \frac{n \cdot K(\Delta x)^2}{2} \quad (n \text{ times})
\]

But \( \Delta x = \frac{b-a}{n} \), so...

\[
\text{Error} \leq \frac{n \cdot K \left( \frac{b-a}{n} \right)^2}{2} = \frac{K(b-a)^2}{2n}
\]

Similar constructions for the other methods give us the following theorem:

**Theorem:** Let \( K_1 = \max |f'(x)| \text{ for } x \in [a,b] \), and \( K_2 = \max |f''(x)| \text{ for } x \in [a,b] \).

Then,

<table>
<thead>
<tr>
<th>Method</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left endpoints</td>
<td>( \leq \frac{K_1 (b-a)^2}{2n} )</td>
</tr>
<tr>
<td>Right endpoints</td>
<td>( \leq \frac{K_1 (b-a)^2}{2n} )</td>
</tr>
<tr>
<td>Midpoints</td>
<td>( \leq \frac{K_2 (b-a)^3}{24n^2} )</td>
</tr>
<tr>
<td>Trapezoids</td>
<td>( \leq \frac{K_2 (b-a)^3}{12n^2} )</td>
</tr>
</tbody>
</table>
Ex: Find an upper bound on the error of estimating \( \int_{0}^{2} \sin x \, dx \) with 10 data points using each of the four methods above.

Solution:

If \( f(x) = \sin x \), then \( f'(x) = \cos x \)
\[ f''(x) = -\sin x. \]

So \( K_1 = \max |\cos x| \) for \( x \in [0,2] \)
\[ K_2 = \max |-\sin x| \] for \( x \in [0,2] \).

Since \( |\cos x| \leq 1 \) and \( |\sin x| \leq 1 \) for all \( x \),
we can take \( K_1 = 1 \) and \( K_2 = 1 \).

1. Left endpoints: error \( \leq \frac{1 \cdot (2-0)^2}{2 \cdot 10} = \frac{1}{5} \)

2. Right endpoints: error \( \leq \frac{1}{5} \) (same as left endpoints)

3. Midpoints: error \( \leq \frac{1 \cdot (2-0)^3}{24 \cdot 10^2} = \frac{1}{300} \)

4. Trapezoids: error \( \leq \frac{1 \cdot (2-0)^3}{12 \cdot 10^2} = \frac{1}{150} \)

Ex: How many data points would be required to estimate \( \int_{0}^{1} e^{x^2} \, dx \) to an accuracy of \( \frac{1}{1000} \) using the right endpoint method? Midpoint method?
Solution: Since we'll be using our error estimates, we should first find $K_1$ and $K_2$.

If $f(x) = e^{x^2}$, then

$$f'(x) = e^{x^2}(x^2)' = 2x e^{x^2}$$

$$f''(x) = 2e^{x^2} + 2x(2x e^{x^2}) = (2+4x^2)e^{x^2}.$$ 

Notice that both $f'(x)$ and $f''(x)$ are $\geq 0$ and increasing on $[0,1]$, so

$$K_1 = \max \left| 2x e^{x^2} \right| = 2(1)e^{(1)^2} = 2e$$

$$K_2 = \max \left| (2+4x^2)e^{x^2} \right| = (2+4(1)^2)e^{(1)^2} = 6e.$$ 

First, let's use right endpoints. We want $n$ so that error $\leq \frac{1}{1000}$.

We know error $< \frac{K_1(b-a)^2}{2n} = \frac{2e(1-0)^2}{2n} = \frac{e}{n}$

and $\frac{e}{n} \leq \frac{1}{1000} \Rightarrow 1000e \leq n$

$\Rightarrow n \geq 2718.28 \quad \text{(ew!)}$

$\Rightarrow n \geq 2719$ 

For midpoints, we have error $< \frac{K_2(b-a)^3}{24n^2} = \frac{6e}{24n^2}$

So $\frac{6e}{24n^2} \leq \frac{1}{1000} \Rightarrow 250e \leq n^2$

$\Rightarrow n \geq \sqrt{250e} \approx 26.06$

$\Rightarrow n \geq 27 \quad \text{(better!)}$