§ 8.4 - The Fundamental Theorem of Calculus.

Here we introduce a powerful theorem that allows us to compute \( \int_a^b f(x) \, dx \) without the limits in § 8.3.

The Fundamental Theorem of Calculus (FTC)

Suppose \( f(x) \) is continuous on \([a, b]\) and let \( F(x) \) be any antiderivative of \( f(x) \) (so \( F'(x) = f(x) \)). Then

\[
\int_a^b f(x) \, dx = F(b) - F(a) = F(x) \bigg|_a^b
\]

\( \text{new notation} \)

So, rather than computing a horrendous limit, we can find the exact area under \( f(x) \) using antiderivatives!!

EX: Find the area under \( f(x) = x^2 + x \) from 0 to 4.

Solution:

\[
\int_0^4 x^2 + x \, dx = \left. \frac{x^3}{3} + \frac{x^2}{2} + C \right|_0^4
\]

\[
= \left( \frac{4^3}{3} + \frac{4^2}{2} + C \right) - \left( \frac{0^3}{3} + \frac{0^2}{2} + C \right) = \frac{88}{3}
\]

Notice that the "+C" terms just cancelled out. This will always happen when computing \( \int_a^b f(x) \, dx \), so...

When computing \( \int_a^b f(x) \, dx \), we don't need +C!
Recap:
\[ \int f(x) \, dx = \text{indefinite integral (general antiderivative)} \]
\[ = F(x) + C \]
\[ \int_{a}^{b} f(x) \, dx = \text{definite integral (area under } f(x) \text{ from } a \text{ to } b) \]
\[ = F(b) - F(a) \]

Properties of the Definite Integral

1. \[ \int_{a}^{a} f(x) \, dx = 0 \]
2. \[ \int_{a}^{b} k \cdot f(x) \, dx = k \cdot \int_{a}^{b} f(x) \, dx \]
3. \[ \int_{a}^{b} f(x) \pm g(x) \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \]
4. \[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]
5. \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \] for any \( c \in \mathbb{R} \)

Important remark: \( \int_{a}^{b} f(x) \, dx \) is actually the signed area “under” \( f(x) \)

- So, if \( f(x) \) is above the \( x \)-axis, then it’s positive
- And if \( f(x) \) is below the \( x \)-axis, then it’s negative.

(all of our examples so far have been above the \( x \)-axis)
Ex:

1. \( \int_0^1 x^3 \, dx = \frac{x^4}{4} \bigg |_0^1 \)
   \[ = \left( \frac{1^4}{4} \right) - \left( \frac{0^4}{4} \right) = \frac{1}{4} \]

2. \( \int_0^\pi \sin x \, dx = -\cos x \bigg |_0^\pi \)
   \[ = (-\cos \pi) - (-\cos 0) \]
   \[ = 1 - (-1) = 2 \]

3. \( \int_0^{2\pi} \sin x \, dx = -\cos x \bigg |_0^{2\pi} \)
   \[ = (-\cos (2\pi)) - (-\cos 0) \]
   \[ = (-1) - (-1) = 0 \] \( \text{Areas cancel!} \)

4. \( \int_{-1}^{1} e^{2t} \, dt = \frac{e^{2t}}{2} \bigg |_{-1}^{1} \)
   \[ = \frac{e^2 - e^{-2}}{2} \]

Important note: If you make a substitution, don’t forget to change \( a \) and \( b \)!!
Example: Calculate \( \int_1^3 x \cos(x^2+1) \, dx \)

Solution: Let \( u = x^2+1 \)
\[ du = 2x \, dx \quad \Rightarrow \quad dx = \frac{du}{2x} \]

When \( x = 1 \), \( u = (1)^2+1 = 2 \)
When \( x = 3 \), \( u = (3)^2+1 = 10 \)

So, \( \int_1^3 x \cos(x^2+1) \, dx = \int_2^{10} x \cos(u) \, \frac{du}{2x} \)
\[ = \frac{1}{2} \int_2^{10} \cos(u) \, du \]
\[ = \frac{1}{2} \sin(u) \bigg|_2^{10} = \frac{1}{2} \left[ \sin(10) - \sin(2) \right] \]

Exercise: Calculate \( \int_1^e \frac{3}{x(1+\ln x)} \, dx \)

Sometimes we may want the area between \( f(x) \) and the x-axis. We do this in 3 steps.

1. Find x-intercepts
2. Integrate \( f(x) \) between each pair of intercepts.
3. Add together these areas in absolute value.

Example: Find the area between \( f(x) = 4-x^2 \) and the x-axis in \([0,4]\).
Solution:

What are the x-intercepts in \([0,4]\)?

\[4-x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2\] (only +2 is in \([0,4]\))

\[
A_1 = \int_{0}^{2} 4-x^2 \, dx = 4x - \frac{x^3}{3} \bigg|_{0}^{2} = \left(4(2) - \frac{(2)^3}{3}\right) - \left(4(0) - \frac{(0)^3}{3}\right) = \frac{16}{3}
\]

\[
A_2 = \int_{2}^{4} 4-x^2 \, dx = 4x - \frac{x^3}{3} \bigg|_{2}^{4} = \left(4(4) - \frac{4^3}{3}\right) - \left(4(2) - \frac{2^3}{3}\right) = \frac{-32}{3}
\]

So Area = \[\frac{16}{3} + \frac{-32}{3} = \frac{16}{3} + \frac{-32}{3} = 48 = \text{[16]}\]