Chapter 8 - Integration

§ 8.1 - Antiderivatives

If we know \( f'(x) \), how can we find \( f(x) \)?

**Definition**: \( F(x) \) is an antiderivative of \( f(x) \) if 
\[
F'(x) = f(x).
\]

**Ex**: An antiderivative of \( f(x) = 4x^3 \) is \( F(x) = x^4 \).

Why? Because \( F'(x) = (x^4)' = 4x^3 \)!

However, this is not the only antiderivative!

For instance, \( G(x) = x^4 + 2 \) and \( H(x) = x^4 - 1 \) are antiderivatives of \( f(x) = 4x^3 \) too.

In general ... 

- If \( F(x) \) and \( G(x) \) are antiderivatives of \( f(x) \), then \( F(x) \) and \( G(x) \) differ by a constant.

**Ex**: \( F(x) = x^2 \), \( G(x) = x^2 + 1 \), and \( H(x) = x^2 - \pi \) are three antiderivatives of \( f(x) = 2x \).

The most general antiderivative of \( f(x) = 2x \) is 
\[
F(x) = x^2 + C \quad (c \in \mathbb{R})
\]

This is called the **indefinite integral** of \( f(x) \).

**Notation**: 
\[
\int f(x) \, dx = F(x) + C
\]

\( \int \) integral of \( f(x) \)

\( dx \) "with respect to \( x" \)
Just like for derivatives, we will now develop some techniques for finding antiderivatives!

**Power Rule**

If $n \neq -1$, then $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

If $n = -1$, then $\int \frac{1}{x} \, dx = \ln |x| + C$

(Why absolute value?? Well... $\frac{1}{x}$ is defined for all $x \neq 0$, but $\ln x$ is defined only for $x > 0$. We can allow $x < 0$ by taking absolute values.)

**Addition, Subtraction, and Constants • Functions**

$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

$\int k \cdot f(x) \, dx = k \cdot \int f(x) \, dx \quad (k = \text{constant})$

**Ex:**

1. $\int 8x^7 \, dx = \frac{8x^8}{8} + C = x^8 + C$

2. $\int x^2 + 1 \, dx = \int x^2 \, dx + \int 1 \, dx = \frac{x^3}{3} + x + C$

3. $\int -x^6 + 3x^3 - x + \pi \, dx = \left[ -\frac{x^7}{7} + \frac{3x^4}{4} - \frac{x^2}{2} + \pi x + C \right]$
Sometimes you may need to rewrite the function so that our rules can be used.

**Ex:**

1. \(\int (x^2 - 1)^2 \, dx = \int x^4 - 2x^2 + 1 \, dx\)
   
   \[= \frac{x^5}{5} - \frac{2x^3}{3} + x + C\]

2. \(\int \sqrt{x} + x^{2/3} \, dx = \int x^{1/2} + x^{2/3} \, dx\)
   
   \[= \frac{x^{3/2}}{3/2} + \frac{x^{5/3}}{5/3} + C\]
   
   \[= \frac{2x^{3/2}}{3} + \frac{3x^{5/3}}{5} + C\]

3. \(\int \frac{x^3 + x + 2}{x^2} \, dx = \int \frac{x^3}{x^2} + \frac{x}{x^2} + \frac{2}{x^2} \, dx\)
   
   \[= \int x + \frac{1}{x} + 2x^{-2} \, dx\]
   
   \[= \frac{x^2}{2} + \ln|x| + 2x^{-1} + C\]
   
   \[= \frac{x^2}{2} + \ln|x| - 2x^{-1} + C\]

**Exponentials:**

1. \(\int e^x \, dx = e^x + C\)

2. \(\int e^{kx} \, dx = \frac{e^{kx}}{k} + C\)

3. \(\int a^x \, dx = \frac{a^x}{\ln(a)} + C\)

4. \(\int a^{kx} \, dx = \frac{a^{kx}}{k \cdot \ln(a)} + C\)
Trig: 1. \[ \int \sin x \, dx = -\cos x + C \]
2. \[ \int \cos x \, dx = \sin x + C \]
3. \[ \int \sec^2 x \, dx = \tan x + C \]
4. \[ \int \csc^2 x \, dx = -\cot x + C \]

Ex:

1. \[ \int \frac{\cos x - \frac{1}{x} + e^{3x}}{x} \, dx = \frac{\sin x - \ln|1| + e^{3x}}{3} + C \]

2. \[ \int 10^x + \sin x - 7^{3x} \, dx = \frac{10^x - \cos x - 7^{3x}}{\ln(10) - 3\ln(7)} + C \]

Note: We can solve for \( C \) if we have extra info.

Ex: Find \( f(x) \) if \( f'(x) = 3x^2 + 2 \) and \( f(0) = 0 \).

Solution: \[ \int 3x^2 + 2 \, dx = x^3 + 2x + C \]

So \( f(x) = x^3 + 2x + C \) for some \( C \in \mathbb{R} \).

\[ f(0) = 0 \implies 0^3 + 2(0) + C = 0 \]

\[ \implies C = 0 \]

So, \( f(x) = x^3 + 2x \)

Ex: Find \( f(x) \) if \( f'(x) = x - \sec^2 x \) and \( f(\pi) = 0 \).

Solution: \[ \int x - \sec^2 x \, dx = \frac{x^2}{2} - \tan x + C \]

Using \( f(\pi) = 0 \), we get \( C = -\frac{\pi^2}{2} \), so \( f(x) = \frac{x^2}{2} - \tan x - \frac{\pi^2}{2} \).