§7.4 - Curve Sketching

Here we'll use our calculus knowledge to accurately sketch the graph of complicated functions.

The Process: To sketch $f(x)$...

1. Find the domain

2. Find the $x$-intercepts $(y=0)$, $y$-intercept $(x=0)$

3. Find vertical asymptotes (check limits when $x \to 0$, $\ln$, $\tan$, etc.)
   Find horizontal asymptotes (check $\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$)

4. Find $f'(x)$ and critical points ($f'(x) = 0$ or DNE)

5. Find $f''(x)$ and find where $f''(x) = 0$ or DNE.

6. Make the table. Test all intervals for increase/decrease, concavity, inflection points, extrema.

7. Plot all interesting points on a graph

8. Connect them as follows:

\begin{align*}
\begin{array}{c|cc|c}
\text{Sign of } f''(x) & - & + & - \\
\hline
\text{Sign of } f'(x) & - & + & - \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c|cc|c}
\hline
+ & & - \\
\hline
& - & + \\
- & + & - \\
\end{array}
\end{align*}
Ex: Use calculus to sketch \( f(x) = x^3 - 6x^2 + 9x \).

Solution:

1. \( f(x) \) is a polynomial, so domain = \( \mathbb{R} \)

2. y-intercept: Set \( x = 0 \)
   \[
   \text{Then } y = 0^3 - 6(0)^2 + 9(0) = 0
   \]
   y-int is \((0,0)\).

   x-intercepts: Set \( y = 0 \)
   \[
   0 = x^3 - 6x^2 + 9x
   = x(x^2 - 6x + 9)
   = x(x-3)^2
   \]
   x-ints are \((0,0), (3,0)\).

3. No vertical asymptotes.

   Horizontal asymptotes? \( \lim_{x \to \infty} f(x) = \infty \), \( \lim_{x \to -\infty} f(x) = -\infty \)

   So no horizontal asymptotes.

4. \( f'(x) = 3x^2 - 12x + 9 \) (exists everywhere)

   \[
   f''(x) = 0 \Rightarrow 3(x^2 - 4x + 3) = 0
   \Rightarrow 3(x-1)(x-3) = 0
   \Rightarrow x = 1 \text{ or } x = 3
   \]

   So \((1,4)\) and \((3,0)\) are critical points.

5. \( f''(x) = 6x - 12 \) (exists everywhere)
\[ f''(x) = 0 \Rightarrow 6x = 12 \]
\[ \Rightarrow x = 2 \]

So (2, 2) is a point of interest.

6. The Table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'' )</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( f' )</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( f )</td>
<td>( \searrow )</td>
<td>( \searrow )</td>
<td>( \nearrow )</td>
</tr>
</tbody>
</table>

Shape

So...

(1, 4) is a local max
(3, 0) is a local min
(2, 2) is an inflection point.

7 & 8.
Ex: Use calculus to sketch \( y = \frac{x^2}{x^2 - 4} \).

Solution:

1. Denominator = 0 when \( x = \pm 2 \), so
   
   \[ \text{Domain} = \{ x \in \mathbb{R} : x \neq \pm 2 \} \]

2. \( y \)-intercept: Set \( x = 0 \)
   
   Then \( y = \frac{0^2}{0^2 - 4} = 0 \)

   \( y \)-int is \((0,0)\)

   \( x \)-intercepts: Set \( y = 0 \)

   Then \( 0 = \frac{x^2}{x^2 - 4} \), so \( x^2 = 0 \).

   \( x \)-int is \((0,0)\).

3. Denominator = 0 when \( x = \pm 2 \), but the numerator is not = 0 here!

   So, we have vertical asymptotes at \( x = -2 \) and \( x = 2 \).

   (In general, there is a vertical asymptote whenever \( f(x) \) has an infinite discontinuity (see §5.2).

   Horizontal asymptotes?

   \[
   \lim_{x \to \infty} \frac{x^2}{x^2 - 4} = \lim_{x \to \infty} \frac{x^2}{x^2} \left( 1 - \frac{4}{x^2} \right) = 1
   \]

   \[
   \lim_{x \to -\infty} \frac{x^2}{x^2 - 4} = 1 \text{ as well} \quad \text{H.A. at } y = 1.\]
Note: A function can have 0, 1, or 2 HA's, so we have to check both \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \).

4. \( f'(x) = \frac{(x^2-4) \cdot 2x - x^2(2x)}{(x^2-4)^2} = -\frac{8x}{(x^2-4)^2} \)

\( f'(x) \) DNE when \( x = \pm 2 \) (but neither does \( f(x) \)) and \( f'(x) = 0 \) when \( x = 0 \).

Critical point: \((0, 0)\).

5. \( f''(x) = \frac{-8(x^2-4)^2 + 8x[(x^2-4)^2]'}{(x^2-4)^4} \)

\[ = -\frac{8(x^2-4)^2 + 8x \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4} \]

\[ = \frac{32x^2 - 8(x^2-4)}{(x^2-4)^3} \]

\[ = \frac{8(3x^2 + 4)}{(x^2-4)^3} \]

So \( f''(x) \) DNE at \( x = \pm 2 \), but \( f''(x) \) is never 0 (why? Because \( 3x^2 + 4 \) is never 0!)

6. The Table:

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'' )</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( f' )</td>
<td>+</td>
<td>+</td>
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<td>( f )</td>
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</tr>
</tbody>
</table>

Shape: \( / \searrow \nearrow \nearrow \nwarrow \nearrow \searrow \)
Our table shows that there is a local max at $x = 0$.

Ex: Use calculus to sketch $y = \frac{2e^x}{1+e^x}$

Solution:

1. Since $e^x > 0$, denominator is never = 0.
   So, domain = IR.

2. $y$-intercept: Set $x = 0$
   Then $y = \frac{2e^0}{1+e^0} = \frac{2}{2} = 1$
   $y$-int is (0, 1).

   $x$-intercepts: Set $y = 0$
   Then $\frac{2e^x}{1+e^x} = 0 \Rightarrow 2e^x = 0$
   $\Rightarrow e^x = 0$ (No Solution)

   So, no $x$-ints.
3. Since denominator \( \neq 0 \), no vertical asymptotes.

Horizontal?

\[
\lim_{x \to \infty} \frac{2e^x}{1+e^x} = \lim_{x \to \infty} \frac{e^x}{e^x} \cdot \left( \frac{2}{\frac{1}{e^x} + 1} \right) = \frac{2}{0+1} = 2
\]

\[
\lim_{x \to -\infty} \frac{2e^x}{1+e^x} = \frac{2(0)}{1+0} = 0
\]

So, we have HA's at \( y = 2 \) and \( y = 0 \).

4. \( f'(x) = \frac{(1+e^x) \cdot 2e^x - 2e^x \cdot e^x}{(1+e^x)^2} \)

\[
= \frac{2e^x}{(1+e^x)^2}
\]

\( f'(x) \) exists everywhere and never = 0.

\implies \text{No critical points!}

5. \( f''(x) = \frac{(1+e^x)^2 \cdot 2e^x - 2e^x \cdot 2e^x [(1+e^x)^2]'}{(1+e^x)^4} \)

\[
= \frac{(1+e^x)^2 - 2e^x - 2e^x \cdot 2(1+e^x) \cdot e^x}{(1+e^x)^4}
\]

\[
= \frac{(1+e^x) \cdot 2e^x - 4e^{2x}}{(1+e^x)^3}
\]

\[
= \frac{2e^x \cdot (1-e^x)}{(1+e^x)^3}
\]

\( f''(x) \) exists everywhere

\( f''(x) = 0 \) when \( 1-e^x = 0 \) (i.e., when \( x = 0 \))

So \((0, 1)\) is a point of interest.
6. The Table:

\[ f'' \quad + \quad - \]
\[ f' \quad + \quad + \]
\[ f \quad \uparrow \quad \cap \quad \uparrow \quad \]

Shape (Inflection point at (0, 1))

7 & 8:

[Graph showing the curve and point (0, 1)]

~ END OF DIFFERENTIAL CALCULUS