§ 7.3 - Higher Derivatives; Concavity

Why stop at just 1 derivative?

The second derivative of \( f(x) \) is \( f''(x) \), given by

\[
f''(x) = (f'(x))'
\]

The third derivative of \( f(x) \) is \( f'''(x) \), given by

\[
f'''(x) = (f''(x))'
\]

After 3, we write \( f^{(4)}(x) \), \( f^{(5)}(x) \), \( f^{(6)}(x) \), etc. for higher derivatives.

**Ex:** If \( f(x) = x^3 - 7x^2 + 2x + 3 \), then

\[
\begin{align*}
f'(x) &= 3x^2 - 14x + 2 \\
f''(x) &= 6x - 14 \\
f'''(x) &= 6 \\
f^{(4)}(x) &= 0 \quad \text{etc.}
\end{align*}
\]

**Ex:** If \( g(x) = \sin x + x \ln x \), then

\[
\begin{align*}
g'(x) &= \cos x + (\ln x + x \cdot x^{-1}) \\
       &= \cos x + \ln x + 1 \\
\end{align*}
\]

\[
\begin{align*}
g''(x) &= -\sin x + \frac{1}{x} \\
g'''(x) &= -\cos x - \frac{1}{x^2} \\
g^{(4)}(x) &= \sin x + \frac{2}{x^3} \quad \text{etc.}
\end{align*}
\]
An Application: Distance, Velocity, Acceleration

Suppose \( f(t) \) = distance travelled at time \( t \).

Then \( f'(t) \) = velocity at time \( t \)
\( f''(t) \) = acceleration at time \( t \)

The higher derivatives have names too!
\( f'''(t) \) = jerk
\( f^{(iv)}(t) \) = jounce (or snap)
\( f^{(v)}(t) \) = crackle
\( f^{(vi)}(t) \) = pop
\( f^{(vii)}(t) \) = lock
\( f^{(viii)}(t) \) = drop. (You don't need to know these.)

Ex: A runner's distance at \( t \) seconds is given by \( f(t) = t^4 + 6t^2 \). What is her acceleration at \( t = 2 \) seconds?

Solution: Velocity = \( f'(t) = 4t^3 - 12t \)

acceleration = \( f''(t) = 12t^2 - 12 \)

At \( t = 2 \), acceleration is \( 12 \cdot 4 - 12 = 36 \text{ m/s}^2 \)
(fast runner!)

Concavity

The second derivative is the rate of change of the first derivative.

So, if \( f''(x) > 0 \), then \( f'(x) \) is increasing, so the slopes of the tangent lines are increasing.
\( f(x) \) would look something like \( \bigcup \) \( \text{Concave up} \).

On the other hand, if \( f''(x) < 0 \), then \( f'(x) \) is decreasing, so \( f(x) \) looks something like \( \bigcap \) \( \text{Concave down} \).

So \( f(x) \) is \textbf{concave up} on an interval \( I \) if \( f''(x) > 0 \).
\textbf{Concave down} on an interval \( I \) if \( f''(x) < 0 \).

A point (in the domain of \( f(x) \)) where concavity changes is called an \textbf{inflection point}.

\[ f(x) \text{ is concave up on } (-\infty, a) \cup (b, \infty), \]
\textbf{concave down} on \( (a, b) \).

\( x = b \) is an inflection point \( (x = a \text{ is NOT (not in domain)}). \)

\textit{Ex:} Find all inflection points and intervals of concavity for \( f(x) = 3x^5 + 5x^4 - 20x^3 + 4 \).

\textit{Solution:} \( f'(x) = 15x^4 + 20x^2 - 60x^2 \)
\( f''(x) = 60x^3 + 60x^2 - 120x \)
\( = 60x(x^2 + x - 2) = 60x(x+2)(x-1) \)
So $f''(x) = 0$ when $x = 0, -2, 1$.
We'll test concavity around these points.

\[ \begin{array}{c|cccc|c}
   & - & + & - & + \\
-2 & & & & \\
0 & & & & \\
1 & & & & \\
f(x) & \bigcup & \bigcup & \bigcup & \bigcup
\end{array} \]

So $f(x)$ is concave up on $(-2, 0) \cup (1, \infty)$, concave down on $(-\infty, -2) \cup (0, 1)$, and $x = -2, x = 0, x = 1$ are inflection pts.

\[ \text{Note: If } x = c \text{ is an inflection point, then either } \]
\[ f''(c) = 0 \text{ or } f''(c) \text{ DNE.} \]

However, just because $f''(c) = 0$ or $f''(c)$ DNE does NOT mean $x = c$ is an inflection point!!

We need to make sure concavity changes!

The second derivative gives us an alternate test for max's and min's.

\[ \text{2nd Derivative Test: If } f''(x) \text{ exists at } x = c \]
\[ \text{and } f'(c) = 0 \text{ (so } x = c \text{ is a critical point) then} \]

1. $f''(c) > 0 \Rightarrow x = c$ is a local min.
2. $f''(c) < 0 \Rightarrow x = c$ is a local max
3. $f''(c) = 0 \Rightarrow$ No information.
Note: The First Derivative Test works just as well. Use whatever test you’d like.

Ex: If \( f(x) = x^2 \), then \( f'(x) = 2x \) and \( f''(x) = 2 \).

Since \( f'(0) = 0 \) and \( f''(0) = 2 > 0 \), \( x = 0 \) is a local min by the 2nd derivative test.

Exercise: Show that \( g(x) = x^4 - 2x^2 \) has a local max at \( x = 0 \) and local mins at \( x = \pm 1 \).

Ex: Find where \( f(x) \) is concave up/down. List any inflection points.

1) \( f(x) = x^4 - 6x^2 \).

Solution: \( f'(x) = 4x^3 - 12x \)
\( f''(x) = 12x^2 - 12 = 12(x-1)(x+1) \) \( (=0 \text{ at } x = \pm 1) \)

\[
\begin{array}{c|ccc}
\text{Test} & f''(x) & + & - & + \\
\hline
& -1 & 0 & 1 & 2 \\
\text{f(x)} & U & \cap & \cup & U
\end{array}
\]

Concave up: \( (-\infty, -1) \cup (1, \infty) \)
Concave down: \( (-1, 1) \)
Inflection pts: \( x = \pm 1 \)

2) \( f(x) = \frac{1}{x} \).

Solution: \( f'(x) = -\frac{1}{x^2} \), \( f''(x) = \frac{2}{x^3} \) \( (\text{never } = 0, \text{ but DNE at } x = 0) \)

\[
\begin{array}{c|ccc}
\text{Test} & f''(x) & - & + \\
\hline
& 0 & 1 & 2 \\
\text{f(x)} & \cap & \cup & U
\end{array}
\]

Concave up: \( (0, \infty) \)
Concave down: \( (-\infty, 0) \)
Inflection pts: None
\( x=0 \) not in domain