Chapter 7 - Applications of Differentiation

Surprise! Derivatives are actually good for something. For many things, in fact. By studying $f'(x)$, we can

- better understand (and accurately graph) $f(x)$.
- solve max/min problems, and
- solve real-world problems involving rates of change.

§7.1 - Increasing/Decreasing Functions

Formally, $f(x)$ is increasing on an interval $I$ if for every $x_1 < x_2$ in $I$, $f(x_1) < f(x_2)$.

Similarly, $f(x)$ is decreasing on an interval $I$ if for every $x_1 > x_2$ in $I$, $f(x_1) > f(x_2)$.

Observations:

- Tangent lines have positive slope when $f(x)$ increasing.
  negative slope when $f(x)$ decreasing.

- At $x=a$, $f'(x)=0$. At $x=b$, $f'(x)$ DNE.
  At $x=c$, $f(x)$ is undefined.
Suppose $f(x)$ is a function and $f'(x)$ exists on an interval $I$.

- If $f'(x) > 0$ for all $x \in I$, then $f(x)$ is increasing on $I$.
- If $f'(x) < 0$ for all $x \in I$, then $f(x)$ is decreasing on $I$.

Ex: If $f(x) = x^2$, then $f'(x) = 2x$.

Notice $f'(x) > 0$ on $(0, \infty)$ and $f'(x) < 0$ on $(-\infty, 0)$.

$\Rightarrow f(x) = x^2$ increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. (seems legit.)

Definition: A critical number (or critical point) of $f(x)$ is a point $c$ in the domain such that either
- $f'(c) = 0$, or
- $f'(c)$ DNE.

The important thing is that a function $f(x)$ can go from increasing to decreasing (or vice versa) only at a critical point, or at a point where $f(x)$ is undefined.

This gives us a slick test for finding where a function is increasing/decreasing!
4 Step Test for Increasing / Decreasing

1. Find $f'(x)$

2. Find all critical points.

3. Plot critical points on a line, as well as any points where $f(x)$ is undefined.

4. Check if $f'(x)$ is $>0$ or $<0$ between these points.
   - $f'(x) > 0 \Rightarrow f(x)$ increasing
   - $f'(x) < 0 \Rightarrow f(x)$ decreasing

Ex: Where is $f(x) = x^3 + 9x^2 - 21x + 4$ increasing/decreasing?

Solution:

1. $f'(x) = 3x^2 + 18x - 21$

2. Note: $f'(x)$ exists everywhere.

   \[
   f'(x) = 0 \Rightarrow 3x^2 + 18x - 21 = 0 \\
   \Rightarrow 3(x^2 + 6x - 7) = 0 \\
   \Rightarrow 3(x + 7)(x - 1) = 0, \text{ so } x = -7 \text{ or } x = 1.
   \]

   The critical points are $x = -7$ and $x = 1$.

3. 

   \[\begin{array}{c|c|c}
   & -7 & 1 \\
   \hline 
   + & - & + \\
   \end{array}\]

   At $x = -8$, $f'(x) > 0$  
   At $x = 0$, $f'(x) < 0$  
   At $x = 2$, $f'(x) > 0$

4. $f(x)$ is increasing on $(-\infty, -7) \cup (1, \infty)$  
   decreasing on $(-7, 1)$. 
Ex: Find where $f(x)$ is increasing/decreasing and list any critical points.

(1) $f(x) = e^{x^2}$.

Solution: Follow the 4-step test.

$$f'(x) = e^{x^2} \cdot (x^2)' = 2x \cdot e^{x^2} \quad \text{(this exists everywhere!)}$$

$$f''(x) = 0 \Rightarrow 2x \cdot e^{x^2} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

(our only critical point)

$$f'(x) \quad - \quad +$$

So $f(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

At $x = -1$, $f'(x) < 0$
At $x = +1$, $f'(x) > 0$

(2) $f(x) = \frac{x-1}{x+2}$

Solution: $f'(x) = \frac{(x-1)(x+2) - (x-1)(x+2)}{(x+2)^2} = \frac{3}{(x+2)^2}$

$f'(x)$ is never 0, but $f'(x)$ DNE at $x = -2$. (however, this is not a critical point: it doesn't belong to the domain!)

$$f'(x) \quad + \quad +$$

So $f(x)$ is increasing on $(-\infty, -2) \cup (-2, \infty)$.

At $x = -3$, $f'(x) > 0$
At $x = 0$, $f'(x) > 0$  

Note: There are no critical points.
(3) \( f(x) = (x+1)^{2/3} \)

Solution: \( f'(x) = \frac{2}{3} (x+1)^{-1/3} = \frac{2}{3(x+1)^{1/3}} \).

\( f'(x) \) is never 0, but \( f'(x) \) DNE at \( x = -1 \).
(This time this is a critical point, as \( x = -1 \) belongs to the domain of \( f(x) \).)

So, our critical point is \( x = -1 \).

\[ \begin{array}{c|c|c} f''(x) & - & + \\ \hline \end{array} \]

So,

\[ \begin{array}{c} f(x) \text{ is increasing on } (-1, \infty) \text{ and} \\ \text{decreasing on } (-\infty, -1) \end{array} \]

At \( x = -2 \), \( f'(x) < 0 \)
At \( x = 0 \), \( f'(x) > 0 \)