§ 5.4 – Definition of the Derivative

The instantaneous rate of change of \( f(x) \) at \( x = a \),

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad (a \in \mathbb{R})
\]

is called the derivative of \( f(x) \) at \( x = a \), and is denoted by \( f'(a) \).

It is also the slope of the tangent line at \( x = a \). This is the line that touches \( f(x) \) at \( x = a \) and at no other nearby points.

We can think of the derivative as a function in its own right:

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

(This is basically what we have above, except now \( x \) is a variable instead of some \( a \in \mathbb{R} \).)

Note: The derivative \( f'(x) \) is sometimes denoted \( \frac{d}{dx} (f(x)) \).
Ex: Find the derivative of \( f(x) \) using the definition.

(Later we will learn faster methods to compute \( f'(x) \), but if you are asked to use the definition, you must do it this way!!)

(1) \( f(x) = x \)

Solution: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

\[
= \lim_{h \to 0} \frac{(x+h) - x}{h} \\
= \lim_{h \to 0} \frac{h}{h} = \boxed{1} \quad (f'(x) \text{ is constant!})
\]

Is this surprising? It shouldn't be. Of course any tangent line for \( f(x) = x \) should have slope 1.

(2) \( f(x) = x^2 \)

Solution: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

\[
= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
= \lim_{h \to 0} \frac{2x + h}{1} = \boxed{2x}
\]
(3) \( f(x) = x^3 \)

Solution: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

\[ = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \]

\[ = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \]

\[ = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \]

\[ = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2 \]

[What do you think \( f'(x) \) is for \( f(x) = x^4 \)? Maybe \( 4x^3 \)? Give it a try!]

(4) \( f(x) = \sqrt{x} \)

Solution: \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

\[ = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \]

\[ = \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \]

\[ = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \]

\[ = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \]

Ex: What is the derivative of \( f(x) = |x| \) at \( x = 0 \)?

Solution: Let's try using the definition!
\[ f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \]
\[ = \lim_{h \to 0} \frac{|0+h| - |0|}{h} \]
\[ = \lim_{h \to 0} \frac{|h|}{h} \quad \text{DNE} \quad (\text{see } \S 5.1) \]

Uhh... Wat?! It doesn't exist?!
This is less surprising when we look at the graph of \( f(x) = 1|x| \).

See, the derivative doesn't exist at sharp points because there are several possible tangent lines!

**Ex:** Find the equation of the tangent line to \( f(x) = x^2 + 3x + 2 \) at \( x = 2 \).

**Solution:** Recall that a line has equation
\[ y = mx + b. \]

\[ M = f'(2) = \lim_{h \to 0} \frac{[(2+h)^2 + 3(2+h) + 2] - [2^2 + 3(2) + 2]}{h} \]
\[ = \lim_{h \to 0} \frac{(4 + 4h + h^2) + (6 + 3h) + 2 - 12}{h} \]
\[ = \lim_{h \to 0} \frac{7h + h^2}{h} = \lim_{h \to 0} (7 + h) = 7 \]

A point on the line \( y = 7x + b \) is \((2, f(2)) = (2, 12)\), so \( 12 = 7(2) + b \Rightarrow b = 12 - 14 = -2 \).

**Tangent line:** \[ y = 7x - 2 \]