§ 5.3 - Rates of Change

While driving down the 401, every 30 minutes you record how far you've travelled:

<table>
<thead>
<tr>
<th>Time</th>
<th>0 hr</th>
<th>0.5 hr</th>
<th>1 hr</th>
<th>1.5 hr</th>
<th>2 hr</th>
<th>2.5 hr</th>
<th>3 hr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance</td>
<td>0 km</td>
<td>55 km</td>
<td>100 km</td>
<td>130 km</td>
<td>200 km</td>
<td>250 km</td>
<td>300 km</td>
</tr>
</tbody>
</table>

Average speed in those 3 hours was

\[
\text{distance} = \frac{300 \text{ km}}{3 \text{ hr}} = 100 \text{ km/hr}.
\]

What was your average speed in the first 1.5 hours?

\[
\text{distance} = \frac{130 \text{ km}}{1.5 \text{ hr}} \approx 86.67 \text{ km/hr}
\]

What was your average speed in the last 1.5 hours?

\[
\text{distance} = \frac{300 \text{ km} - 130 \text{ km}}{1.5 \text{ hr}} = \frac{170 \text{ km}}{1.5 \text{ hr}} \approx 113.33 \text{ km/hr}
\]

Get the idea? Given \( f(x) \), the average rate of change between \( x = a \) and \( x = b \) is

\[
\frac{f(b) - f(a)}{b - a}.
\]
Ex: A runner's distance (in metres) over 60 seconds is given by
\[ f(t) = t + 0.1t^2. \]

(1) What is the runner's average speed over these 60 seconds?

\[
\text{Solution: } \frac{f(60) - f(0)}{60 - 0} = \frac{60 + 0.1(60)^2}{60} = [7 \text{ m/s}] 
\]

(2) What is the average speed over the last 10 seconds?

\[
\text{Solution: } \frac{f(60) - f(50)}{60 - 50} = \frac{[60 + 0.1(60)^2] - [50 + 0.1(50)^2]}{10} = [12 \text{ m/s}] 
\]

(3) The last 1 second?

\[
\text{Solution: } \frac{f(60) - f(59)}{60 - 59} = \frac{[60 - 0.1(60)^2] - [59 + 0.1(59)^2]}{1} \approx [12.9 \text{ m/s}] 
\]

(4) The last millisecond? (0.001th of 1 second)

\[
\text{Solution: } \frac{f(60) - f(59.999)}{60 - 59.999} \approx 12.999901 \text{ m/s} 
\]

(5) What was the runner's instantaneous speed at \( t = 60 \) s?

We guess 13 m/s. To be sure, we need... LIMITS!!
The instantaneous rate of change of \( f(x) \) at \( x = a \) is
\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

Back to our example... the instantaneous speed at \( t = 60 \) seconds is
\[
\lim_{h \to 0} \frac{f(60+h) - f(60)}{h} = \lim_{h \to 0} \frac{(60+h) + 0.1(60+h)^2 - (60+0.1(60)^2)}{h} = \lim_{h \to 0} \frac{60+h+360+12h+0.1h^2 - 60-360}{h} = \lim_{h \to 0} \frac{13h+0.1h^2}{h} = \lim_{h \to 0} 13 + 0.1h = 13 \text{ m/s}
\]

Ex: Find the instantaneous rate of change for \( f(x) = x^2 + 3x \) at \( x = 1 \)

Solution:
\[
\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{((1+h)^2 + 3(1+h)) - (1^2 + 3(1))}{h} = \lim_{h \to 0} \frac{(1+2h+h^2) + (3+3h) - 4}{h} = \lim_{h \to 0} \frac{5h+h^2}{h} = \lim_{h \to 0} 5 + h = 5
\]
(2) \( x = 2 \)

Solution: \[ \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{[(2+h)^2 + 3(2+h)] - [2^2 + 3(2)]}{h} \]

\[ = \lim_{h \to 0} \frac{(h^2 + 4h + h^2) + (6 + 3h) - 10}{h} \]

\[ = \lim_{h \to 0} \frac{7h + h^2}{h} \]

\[ = \lim_{h \to 0} \frac{7h}{h} + \lim_{h \to 0} h = 7 \]

(3) \( x = a \)

Solution: \[ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[(a+h)^2 + 3(a+h)] - [a^2 + 3a]}{h} \]

\[ = \lim_{h \to 0} \frac{(a^2 + 2ah + h^2) + (3a + 3h) - a^2 - 3a}{h} \]

\[ = \lim_{h \to 0} \frac{2ah + 3h + h^2}{h} \]

\[ = \lim_{h \to 0} \frac{2ah}{h} + \lim_{h \to 0} h = 2a + 3 \]

Congratulations! You've just computed your first derivative!