§5.2 - Continuity

Intuition: A function is continuous if you can draw its graph without lifting your pen.

Ex: Polynomials, logarithms, exponentials, \( \sin x, \cos x \) are all continuous.

Formally: A function \( f(x) \) is continuous at \( x = a \) if:

1. \( f(a) \) exists
2. \( \lim_{x \to a} f(x) \) exists, and
3. \( \lim_{x \to a} f(x) = f(a) \) (We often just write 3.)

If \( f(x) \) is continuous at every point in an interval \( I \), we say that \( f(x) \) is continuous on \( I \).

Ex: Where is this function continuous?

Where is it discontinuous?

It is discontinuous at:
- \( x = -3 \) (limit DNE)
- \( x = 1 \) (limit DNE)
- \( x = 3 \) (\( f(3) \) DNE)
- \( x = 4 \) (limit exists, but doesn't equal \( f(4) \)!

It is continuous everywhere else in \([-5, 5]\).
There are 3 types of discontinuity.

1. If \( \lim_{x \to a} f(x) \) exists but
   is not equal to \( f(a) \), it is a removable discontinuity.

   (So either \( f(a) \) DNE or it is just different from \( \lim_{x \to a} f(x) \))

2. If both \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \)
   exist and are finite, but not equal, it is a jump discontinuity.

3. If \( \lim_{x \to a^-} f(x) = \pm \infty \) or \( \lim_{x \to a^+} f(x) = \pm \infty \)
   it is an infinite discontinuity.

Refer back to the function from the 1st example.

What types of discontinuities are present?
Solution:

- $\lim_{x \to -3^-} f(x) = -\infty \Rightarrow$ infinite discontinuity at $x = -3$.

- $\lim_{x \to 1^-} f(x) = 2$ and $\lim_{x \to 1^+} f(x) = 1$. The limits are finite but not equal $\Rightarrow$ jump discontinuity at $x = 1$.

- $\lim_{x \to 3^-} f(x) = 1$ and $\lim_{x \to 3^+} f(x) = 1$ (equal) but $f(3)$ DNE. $\Rightarrow$ removable discontinuity at $x = 3$.

- $\lim_{x \to 4^-} f(x) = 1$ and $\lim_{x \to 4^+} f(x) = 1$ (equal) but $f(4) \neq 1$. $\Rightarrow$ removable discontinuity at $x = 4$.

Ex: Sketch $f(x) = \begin{cases} 
-2 & \text{if } x < -2 \\
3 & \text{if } x = -2 \\
x+1 & \text{if } -2 < x \leq 0 \\
x^2-1 & \text{if } x > 0
\end{cases}$

Where is $f(x)$ continuous? What types of discontinuities does $f(x)$ possess?

Solution:

$\Rightarrow \lim_{x \to -2^-} f(x) = -2$, $\lim_{x \to -2^+} f(x) = -1$

Finite but not equal, so jump discontinuity at $x = -2$.

$\Rightarrow \lim_{x \to 0^-} f(x) = 1$, $\lim_{x \to 0^+} f(x) = -1$

Finite but not equal, so jump discontinuity at $x = 0$.

$\Rightarrow$ Elsewhere, $f(x)$ is continuous.