Chapter 5 - Calculus Begins (Finally!)

3 main topics

- **Limits** (What does \( f(x) \) do as \( x \) approaches a certain value?)
- **Continuity** (Is \( f(x) \) well-behaved or does it jump around?)
- **Derivatives** (How "steep" is \( f(x) \) at a point \( x \)?)

§5.1 - Limits

Consider the following 4 functions:

![Graphs of f₁(x), f₂(x), f₃(x), f₄(x)]

Note: \( f₁ \) and \( f₃ \) are defined at \( x = 1 \), while \( f₂ \) and \( f₄ \) are not.

Nevertheless, we can still talk about the behaviour of each function as \( x \) gets infinitely close to 1!

As \( x \to 1 \)...

- \( f₁(x) \) approaches 2
- \( f₂(x) \) approaches 2
- \( f₃(x) \) approaches 2 if \( x \) comes from the left
- \( f₄(x) \) approaches \(-\infty\) if \( x \) comes from the right
If \( f(x) \) approaches a finite number \( L \) as \( x \) gets infinitely close to \( a \) but not equal to \( a \), we say

"the limit as \( x \) approaches \( a \) of \( f(x) \) is \( L \)."

and write \( \lim_{x \to a} f(x) = L \)

Note: \( L \) must be the same if \( x \) comes from the right or the left!

These limits are denoted by

\[
\begin{align*}
\lim_{x \to a^-} f(x) & \quad (x \to a \text{ from left}) \\
\lim_{x \to a^+} f(x) & \quad (x \to a \text{ from right})
\end{align*}
\]

If \( f(x) \) does not approach a finite value, or if the left/right limits are different, we say

"the limit as \( x \) approaches \( a \) of \( f(x) \) does not exist" (DNE).

Ex: Let \( f_1, f_2, f_3, f_4 \) be as before.

- \( \lim_{x \to 1^-} f_1(x) = 2 \)
- \( \lim_{x \to 1^+} f_2(x) = 2 \)
- \( \lim_{x \to 1^-} f_3(x) = 2 \) while \( \lim_{x \to 1^+} f_3(x) = -2 \) \( \neq \lim_{x \to 1^-} f_3(x) \) DNE
- \( \lim_{x \to 1} f_4(x) \) DNE \( (f(x) \to -\infty \ldots \text{NOT FINITE!}) \)
Example:
\[ f(x) = \begin{cases} 
  x + 1 & \text{if } x \neq 2 \\
  0 & \text{if } x = 2 
\end{cases} \]

Then \( f(2) = 0 \) but \( \lim_{x \to 2} f(x) = 3 \).

(we care about what happens near \( x = 2 \), not at \( x = 2 \))

Example:
\[ f(x) = \begin{cases} 
  -x & \text{if } x < 0 \\
  x^2 & \text{if } 0 \leq x < 1 \\
  2 & \text{if } x \geq 1 
\end{cases} \]

Then \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0 \) \quad \& \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0 \)

\( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} x^2 = 0 \) \quad \text{equal, so } \lim_{x \to 0} f(x) = 0. \)

But \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1 \) \quad \text{not equal, so } \lim_{x \to 1} f(x) \text{ DNE.} \)

\( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2 \)

We also consider limits at \( \pm \infty \), which correspond to horizontal asymptotes of \( f(x) \).

Example:
\[ \lim_{x \to \infty} \frac{1}{x} = 0 \] (denominator becomes huge, so fraction becomes tiny!)

\[ \lim_{x \to -\infty} \frac{1}{x} = 0 \]  \quad \text{We'll use these facts often.}
Limit Rules.

Suppose \( \lim_{x \to a} f(x) = F \) and \( \lim_{x \to a} g(x) = G \).

1. \( \lim_{x \to a} f(x) \pm g(x) = F \pm G \)

2. \( \lim_{x \to a} f(x) \cdot g(x) = F \cdot G \)

3. \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{F}{G} \) (provided \( G \neq 0 \))

4. \( \lim_{x \to a} c \cdot f(x) = c \cdot F \)

5. \( \lim_{x \to a} (f(x))^k = F^k \) (provided limit exists)

6. \( \lim_{x \to a} b^{f(x)} = b^F \) (\( b > 0 \))

7. \( \lim_{x \to a} \log_b(f(x)) = \log_b(F) \) (\( F > 0 \))

8. \( \lim_{x \to a} \sin(f(x)) = \sin(F) \)

\( \lim_{x \to a} \cos(f(x)) = \cos(F) \),

i.e., just plug in \( a \)

Remark: Rules 1., 4., and 5. imply that \( \lim_{x \to a} f(x) = f(a) \) when \( f(x) \) is a polynomial!

Ex: \( \lim_{x \to 1} x^2 - 2x + 1 = (1)^2 - 2(1) + 1 = 0 \)

\( \lim_{x \to 0} \sqrt{x^2 + 3} = \sqrt{(0)^2 + 3} = \sqrt{3} \)

\( \lim_{x \to \pi/2} \cos(2x) = \cos(2(\pi/2)) = \cos(\pi) = -1 \)
Ex: \( \lim_{x \to 0} e^{x+1} - \ln (\sin(x) + 1) = e^{0+1} - \ln (\sin(0) + 1) = e - \ln (1) = e \)

But of course the story doesn't end here...

Ex: Find the limit if it exists.

1. \( \lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5} \)
2. \( \lim_{x \to \infty} \frac{3x^2 + 1}{5x^2 + 2x - 1} \)
3. \( \lim_{x \to -\infty} \frac{8x + 2}{2x^2 - 5} \)
4. \( \lim_{x \to \frac{\pi}{2}} \frac{\sin(2x)}{\cos x} \)
5. \( \lim_{x \to 25} \frac{\sqrt{x^2 - 5}}{x - 25} \)
6. \( \lim_{x \to 0} \frac{|x|}{x} \)

None of these limits can be evaluated by simply "plugging in \( a \)"... more work must be done.

Strategies for Finding Limits

Step 1: Can we evaluate the limit by plugging in \( a \)?

Does it have the form \( \# \cdot \infty \) (=\( \infty \)), \( \# \) (=\( 0 \)),
or \( \# \) (=\( \pm \infty \)) ?

Step 2: Is it an indeterminate form? 

(\( 0/0, \; \pm \infty/\infty, \; 0^\infty, \; \infty^0, \; 1^\infty, \; 0 \cdot \infty, \; \infty - \infty \), etc.)
Try... • factoring and cancelling
• rationalizing denominator or numerator
• using trig. identities.
• checking left/right limits.
• factoring highest power of x in numerator and denominator.

Step 3: After each modification in Step 2, try to evaluate the limit again.

Okay! Let’s try (1) - (6) from previous example.

Solution:

1. \[ \lim_{x \to 5} \frac{x^2 - 3x - 10}{x - 5} = \lim_{x \to 5} \frac{(x+2)(x-5)}{x-5} = \lim_{x \to 5} x + 2 = 7 \]

2. \[ \lim_{x \to \infty} \frac{3x^2 + 1}{5x^2 + 2x - 1} = \lim_{x \to \infty} \frac{x^2 (3 + \frac{1}{x^2})}{x^2 (5 + \frac{2}{x} - \frac{1}{x^2})} = \lim_{x \to \infty} \frac{3 + \frac{1}{x^2}}{5 + \frac{2}{x} - \frac{1}{x^2}} = \frac{3 - 0}{5 + 0 - 0} = \frac{3}{5} \]

3. \[ \lim_{x \to -\infty} \frac{8x + 2}{2x^2 - 5} = \lim_{x \to -\infty} \frac{x (8 + \frac{2}{x})}{x^2 (2 - \frac{5}{x^2})} = \lim_{x \to -\infty} \frac{1}{x} \left( \frac{8 + \frac{2}{x}}{2 - \frac{5}{x^2}} \right) \]

\[ \text{Goes to 0} \quad \text{Goes to} \quad \frac{8 + 0}{2 - 0} = \frac{8}{2} = 4. \]

= 0
(4) \[
\lim_{x \to \pi/2} \frac{\sin(2x)}{\cos x} = \lim_{x \to \pi/2} \frac{2 \sin x \cdot \cos x}{\cos x} \\
= \lim_{x \to \pi/2} 2 \sin x \\
= 2 \cdot \sin(\pi/2) = 2
\]

(5) \[
\lim_{x \to 25} \frac{\sqrt{x - 5}}{x - 25} = \lim_{x \to 25} \frac{\sqrt{x - 5} \cdot \frac{\sqrt{x + 5}}{\sqrt{x + 5}}}{x - 25} \\
= \lim_{x \to 25} \frac{x - 25}{(x - 25)(\sqrt{x + 5})} \\
= \lim_{x \to 25} \frac{1}{\sqrt{x + 5}} \\
= \frac{1}{\sqrt{25 + 5}} = \frac{1}{10}
\]

(6). Recall that \( |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases} \)

For \( \lim_{x \to 0} \frac{|x|}{x} \), we'll check the left & right limits.

\[
\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} -x = -1 \\
\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} x = 1
\]

Not equal, so \( \lim_{x \to 0} \frac{|x|}{x} \) DNE

The Squeeze Theorem

If \( f(x), g(x), h(x) \) are functions, and
\( f(x) \leq g(x) \leq h(x) \) around \( A \), and if
\( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \),
then \( \lim_{x \to a} g(x) = L \).
Note: Good for limits involving $\sin x$ and $\cos x$, as $-1 \leq \sin x \leq 1$, $-1 \leq \cos x \leq 1$.

Ex: Find $\lim_{x \to \infty} \frac{\sin x}{x^2}$.

Solution: Since $-1 \leq \sin x \leq 1$, we have

\[ -\frac{1}{x^2} \leq \frac{\sin x}{x^2} \leq \frac{1}{x^2}. \]

Thus, $\lim_{x \to \infty} -\frac{1}{x^2} \leq \lim_{x \to \infty} \frac{\sin x}{x^2} \leq \lim_{x \to \infty} \frac{1}{x^2} \Rightarrow 0 \leq \lim_{x \to \infty} \frac{\sin x}{x^2} \leq 0$.

$\Rightarrow \lim_{x \to \infty} \frac{\sin x}{x^2} = 0$ by the squeeze theorem.

Exercise: Find $\lim_{x \to 0} x^2 \sin \left(\frac{1}{x}\right)$.