§ 4.4 - Trigonometric Functions

Recall: SOH CAH TOA

\[
\begin{align*}
\sin \theta &= \frac{O}{H} \\
\cos \theta &= \frac{A}{H} \\
\tan \theta &= \frac{O}{A} \\
\csc \theta &= \frac{1}{\sin \theta} \\
\sec \theta &= \frac{1}{\cos \theta} \\
\cot \theta &= \frac{1}{\tan \theta}
\end{align*}
\]

In calculus, the angle \( \theta \) is always measured in radians.

1 radian = angle that cuts off arc length equal to the radius

To convert degrees \( \leftrightarrow \) radians, use

\[
\begin{align*}
\text{degrees} &= \frac{\text{radians} \cdot 180}{\pi} \\
\text{radians} &= \frac{\text{degrees} \cdot \pi}{180}
\end{align*}
\]

Ex: degrees 0 30 45 60 90 180 360
radians 0 \( \frac{\pi}{6} \) \( \frac{\pi}{4} \) \( \frac{\pi}{3} \) \( \frac{\pi}{2} \) \( \pi \) 2\( \pi \)

Most values of \( \cos \theta \) and \( \sin \theta \) are found with a calculator. The ones we should know by heart are found on the unit circle.
What about \( \tan \Theta \)?

**Note:** \( \tan \Theta = \frac{\sin \Theta}{\cos \Theta} \)

If we know \( \sin \Theta \) and \( \cos \Theta \), we can calculate \( \tan \Theta \).

**Ex:** \( \tan \left( \frac{\pi}{3} \right) = \frac{\sin \left( \frac{\pi}{3} \right)}{\cos \left( \frac{\pi}{3} \right)} = \frac{\sqrt{3}}{\frac{1}{2}} = \sqrt{3} \)

\( \tan \left( \frac{\pi}{4} \right) = \frac{\sin \left( \frac{\pi}{4} \right)}{\cos \left( \frac{\pi}{4} \right)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \)

\( \tan \left( \frac{\pi}{6} \right) = \frac{\sin \left( \frac{\pi}{6} \right)}{\cos \left( \frac{\pi}{6} \right)} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \)

\( \tan (0) = \frac{\sin (0)}{\cos (0)} = 0 = \frac{0}{1} \)

Since \( \cos \left( \frac{\pi}{2} \right) = 0 \), \( \tan \left( \frac{\pi}{2} \right) \) is undefined.

We can remember where \( \sin \Theta, \cos \Theta, \tan \Theta \) are + or - by using the CAST rule:

- **S** for \( \sin \Theta \)
- **A** for \( \cos \Theta \)
- **T** for \( \tan \Theta \)
- **C** for \( \cot \Theta \)

**Trig Identities:** There are lots.

Most important: \( \sin^2 \Theta + \cos^2 \Theta = 1 \)

Divide \( \Theta \) by \( \sin^2 \Theta \) to get \( 1 + \cot^2 \Theta = \csc^2 \Theta \)

Divide \( \Theta \) by \( \cos^2 \Theta \) to get \( \tan^2 \Theta + 1 = \sec^2 \Theta \).
Sum/Difference of angles:

\[
\sin(x+y) = \sin x \cos y + \cos x \sin y
\]
\[
\sin(x-y) = \sin x \cos y - \cos x \sin y
\]
\[
\cos(x+y) = \cos x \cos y - \sin x \sin y
\]
\[
\cos(x-y) = \cos x \cos y + \sin x \sin y
\]

Double angle:

\[
\sin 2\theta = 2 \sin \theta \cos \theta
\]
\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta
\]
From these we get
\[
\sin^2 \theta = \frac{1 - \cos 2\theta}{2}
\]
\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
\]

Graphs of Trig Functions.

\[y = \sin x\]

\[y = \cos x\]

\[y = \tan x\]
A periodic function is a function \( f(x) \) such that

\[
 f(x) = f(x+a)
\]

for some positive number \( a \) (called the period of \( f \)).

Notice from the graphs: \( \sin \theta, \cos \theta \) are periodic with period \( 2\pi \).
\( \tan \theta \) is periodic with period \( \pi \).

We also note that \( \sin \theta \) and \( \cos \theta \) reach a height of 1. This is called the **amplitude**.

In general, if \( y = A \cdot \sin(B(x-C)) + D \)

\[
\text{or } y = A \cdot \cos(B(x-C)) + D
\]

then \( A = \text{amplitude} \) \hspace{1cm} \text{Period} = \frac{2\pi}{B}

\( C = \text{horizontal shift} \) \hspace{1cm} \text{D = vertical shift}.

**Ex:** \( y = 2\sin 2x \) has amplitude = 2 and period = \( \frac{2\pi}{2} = \pi \)

(no horizontal/vertical shifts)

![Graph of sin and 2sin2x functions](image)

**Solving Trig Equations**

**Ex:** Solve for \( \theta \):

1. \( \sin 2\theta = \cos \theta \)
2. \( 4 \cos \theta = 4 + \sin^2 \theta \).
Solution: (1) Write $\sin 2\theta = 2 \sin \theta \cos \theta$ to get

\[
2 \sin \theta \cos \theta = \cos \theta \Rightarrow 2 \sin \theta \cos \theta - \cos \theta = 0
\]

\[
\Rightarrow \cos \theta (2 \sin \theta - 1) = 0
\]

So, $\cos \theta = 0$

OR $2 \sin \theta - 1 = 0$ (so $\sin \theta = \frac{1}{2}$)

$\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$

$\sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$

So,

\[
\theta = \left\{ \begin{array}{l}
\frac{\pi}{2} + 2k\pi \quad (k=0, \pm1, \pm2, \ldots) \\
\frac{3\pi}{2} + 2k\pi \quad (k=0, \pm1, \pm2, \ldots) \\
\frac{\pi}{6} + 2k\pi \quad (k=0, \pm1, \pm2, \ldots) \\
\frac{5\pi}{6} + 2k\pi \quad (k=0, \pm1, \pm2, \ldots)
\end{array} \right.
\]

Basically, find all $\theta$ in $[0, 2\pi)$ and then account for repeats from periodicity!

(2) $4 \cos \theta = 4 + \sin^2 \theta$

\[
= 4 + (1 - \cos^2 \theta)
\]

\[
= 5 - \cos^2 \theta
\]

\[
\Rightarrow \cos^2 \theta + 4 \cos \theta - 5 = 0
\]

\[
\Rightarrow (\cos \theta + 5)(\cos \theta - 1) = 0
\]

So $\cos \theta = -5$ (IMPOSSIBLE! $-1 \leq \cos \theta \leq 1$)

or $\cos \theta = 1$, so $\theta = 0, 2\pi, 4\pi, 6\pi, \ldots$

We have $\theta = 2k\pi, \quad k=0, \pm1, \pm2, \ldots$

Sine and Cosine Laws

Sine Law: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Cosine Law: $c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$
Ex: Solve for the unknown:

(1) \[ \triangle \quad 3 \quad \frac{\pi}{3} \quad 8 \quad x \]

(2) \[ \triangle \quad \frac{\pi}{6} \quad 4 \quad x \]

Solution:

(1) We use cosine law:

\[ x^2 = 3^2 + 8^2 - 2(3)(8) \cos(\frac{\pi}{3}) \]

\[ = 9 + 64 - 48 \left(\frac{1}{2}\right) \]

\[ = 73 - 24 \]

\[ = 49 \]

\[ \Rightarrow x = 7 \]

(2) We use sine law:

\[ \frac{\sin X}{6} = \frac{\sin(\frac{\pi}{6})}{4} \Rightarrow \sin X = \frac{6 \cdot \left(\frac{1}{2}\right)}{4} = \frac{3}{4} \]

\[ \Rightarrow X = \sin^{-1}\left(\frac{3}{4}\right) \]

Note: \( \sin X \) achieves a value of \( \frac{3}{4} \) once in quadrant I and once in quadrant II. Both are valid solutions!

With a calculator, we get

\[ X = \sin^{-1}\left(\frac{3}{4}\right) \approx 0.848 \text{ radians} \quad \text{(quadrant I solution)} \]

or

\[ X = \pi - \sin^{-1}\left(\frac{3}{4}\right) \approx 2.29 \text{ radians} \quad \text{(quadrant II solution)} \]