§4.3 - Growth and Decay

Many real life processes follow an exponential growth/decay model:

\[ A(t) = A_0 e^{k \cdot t} \]

- \( A(t) \) = amount of substance at time \( t \).
- \( A_0 \) = initial amount (amount at time \( t = 0 \)).
- \( k \) = constant (\( k > 0 \) for exp growth, \( k < 0 \) for exp. decay).

Typical problem:
- Find \( k \) given some initial data
- Use \( k \) to find amount of substance at some other time.

**Ex [Population Growth]:** A rabbit colony starts with 100 rabbits. After 3 years, it has 900 rabbits. How many will it have after 6 years?

**Solution:** We are given \( A_0 = 100 \), \( A(3) = 900 \).

Our model is

\[ A(t) = 100 e^{k \cdot t} \]

To find \( A(6) \), we should first determine \( k \) by using \( A(3) = 900 \).

\[ A(3) = 900 \Rightarrow 900 = 100 e^{3k} \]
\[ 9 = e^{3k} \]
\[ \ln(9) = \ln(e^{3k}) = 3k \] (\( \ln \) both sides)
\[ k = \frac{\ln(9)}{3} \]

The model is

\[ A(t) = 100 e^{\frac{\ln(9)}{3} \cdot t} \]
Thus, \[ A(6) = 100 e^{\frac{\ln(9)}{3} \cdot 6} = 100 e^{2 \cdot \ln(9)} = 100 e^{\ln(9^2)} = 100 \cdot 9^2 = \frac{8100}{700} \text{ rabbits (wow!)} \]

**[Radioactive Decay]**: A radioactive isotope has a half-life of 10 years. How much of this substance will be left after 23 years?

**Solution**: What is \( A_0 \)?

We'll say \( A_0 = 100 \) (percent), so \( A(10) = 50 \) (after 10 years, only half is left!)

Our model is \( A(t) = 100 e^{k \cdot t} \).

Using \( A(10) = 50 \), we can solve for \( k \):

\[
\begin{align*}
A(10) = 50 & \Rightarrow 50 = 100 e^{k \cdot 10} \\
& \Rightarrow 1 = e^{10k} \\
& \Rightarrow \ln(\frac{1}{2}) = \ln(e^{10k}) = 10k \quad (\ln \text{ both sides})
\end{align*}
\]

So, \( k = \frac{\ln(\frac{1}{2})}{10} \).

Our model is \( A(t) = 100 e^{\frac{\ln(\frac{1}{2})}{10} \cdot t} \).

This means that \( A(23) = 100 e^{\frac{\ln(\frac{1}{2})}{10} \cdot 23} \approx 20.31 \)

Thus, \( \approx 20.31\% \) of the substance is left after 23 years.
A slightly different example: the amount of chemical that will dissolve in a solution increases exponentially as temperature increases!

**Ex [Chemical Dissolution]:** At $0^\circ$C, 1000g of chemical dissolves in a solution. At $10^\circ$C, 1100g dissolves. At what temperature will 1500g dissolve?

**Solution:** We are given \[
\begin{align*}
A_0 &= 1000 \\
A(10) &= 1100
\end{align*}
\]

Our model is $A(t) = 1000 e^{kt}$ (now, $t =$ temperature).

We'll solve for $K$ using $A(10) = 1100$.

$$A(10) = 1100 \Rightarrow 1100 = 1000 e^{k \cdot 10}$$

$$\Rightarrow \frac{11}{10} = e^{10k}$$

$$\Rightarrow \ln \left( \frac{11}{10} \right) = \ln(e^{10k}) = 10k$$

So $k = \frac{\ln \left( \frac{11}{10} \right)}{10}$

Our model is $A(t) = 1000 e^{\frac{\ln \left( \frac{11}{10} \right)}{10} \cdot t}$.

We would like to know $t$ such that $A(t) = 1500$.

Solve for $t$: \[
1500 = 1000 e^{\frac{\ln \left( \frac{11}{10} \right)}{10} \cdot t}
\]

$$\Rightarrow \frac{3}{2} = e^{\frac{\ln \left( \frac{11}{10} \right)}{10} \cdot t}$$

$$\Rightarrow \ln \left( \frac{3}{2} \right) = \frac{\ln \left( \frac{11}{10} \right)}{10} \cdot t \Rightarrow t = \frac{10 \ln \left( \frac{3}{2} \right)}{\ln \left( \frac{11}{10} \right)} \approx 42.5^\circ C$$