§ 3.4 - Quadratic Functions; Translations & Reflections

Recall: A quadratic function has the form

\[ y = ax^2 + bx + c \quad (a, b, c \in \mathbb{R}, \ a \neq 0) \]

Above: Graphs of quadratic functions, called parabolas. The "peak/trough" point is the vertex.

A parabola is symmetric about the vertical line passing through its vertex.

Completing the Square

To graph a parabola, we first write it as

\[ y = a(x-h)^2 + k \]

- The point \((h,k)\) is the vertex.
- The parabola opens upward \(\uparrow\) if \(a > 0\).
  opens downward \(\downarrow\) if \(a < 0\)

The process of converting \(ax^2+bx+c\) to the above form is called completing the square.
3 steps in completing the square.

Ex: Complete the square for \( y = 2x^2 - 4x + 3 \).

(1) Factor the coefficient on \( x^2 \) from all \( x \) terms.

You'll get something like \( y = a(x^2 + px) + q \).

\[ \Rightarrow y = 2(x^2 - 2x) + 3 \]

(so \( a = 2, \ p = -2, \ q = 3 \)).

(2) Add and subtract \( \left( \frac{p}{2} \right)^2 \) within the brackets.

Add and subtract \( \left( \frac{p}{2} \right)^2 = 1 \).

\[ y = 2(x^2 - 2x + 1 - 1) + 3 \]

\[ \Rightarrow y = 2(x^2 - 2x + 1) - 2 + 3 \]

\[ \Rightarrow y = 2(x^2 - 2x + 1) + 1 \].

(3) Factor the bracketed term as \( (x + \frac{p}{2})^2 \).

\[ \frac{p}{2} = -1, \ so \]

\[ y = 2(x - 1)^2 + 1 \]

You're done!

Yay!

Ex: Complete the square for \( y = -x^2 + 4x + 5 \).
Solution: Just follow the 3 steps.

(1) \[ y = -(x^2 - 4x) + 5 \] (factor coefficient)

(2) \[ y = -(x^2 - 4x + 4 - 4) + 5 \] (add and subtract)

\[ \Rightarrow y = -(x^2 - 4x + 4) + 4 + 5 \]

(3) \[ y = -(x - 2)^2 + 9 \] (factor brackets)

Graphing a Quadratic

To graph \[ y = a(x-h)^2 + k \] ...

- plot vertex at \((h, k)\)
- plot \(y\)-intercept (found by setting \(x=0\))
- plot \(x\)-intercept(s) (found by setting \(y=0\))
- Connect points to make \(\uparrow\) if \(a>0\), \(\downarrow\) if \(a<0\).

Note: The parabola may not have \(x\)-intercepts!
If \(a\cdot k > 0\), then there are none.

Ex: Graph the quadratic \[ y = 2(x-1)^2 + 1. \]

Solution: Start with the vertex.
It is at \((h, k) = (1, 1)\).
y-intercept? Set \( x = 0 \).

\[
y = 2(0-1)^2 + 1 = 2(1) + 1 = 3
\]

So, y-int is \((0, 3)\).

x-intercept(s)? There are none! \( a \cdot k = 2 \cdot 1 = 2 > 0 \)

If we had tried setting \( y = 0 \) and solving for \( x \), we would get \((x-1)^2 = -\frac{1}{2}\). We can't square root \(-\frac{1}{2}\)!

Ex: Graph the quadratic \( y = -(x-2)^2 + 9 \)

Solution: Vertex = \((2, 9)\)

Since \( a = -1 < 0 \), parabola opens down \( \swarrow \).

y-intercept? Set \( x = 0 \).

\[
y = -(0-2)^2 + 9 = -4 + 9 = 5
\]

\( \Rightarrow \) y-int is \((0, 5)\).

x-intercept(s)? \( a \cdot k = -9 \), so there are x-ints!

Set \( y = 0 \) and solve.
\[ y = 0 \Rightarrow 0 = -(x-2)^2 + 9 \]
\[ \Rightarrow (x-2)^2 = 9 \]
\[ \Rightarrow x-2 = \pm 3 \]

So \( x = -1 \) or \( x = 5 \). The \( x \)-intercepts are \((-1,0), (5,0)\).

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**Translations & Reflections**

Suppose \( f(x) \) is a function with graph →

Let \( c > 0 \) be a constant

\( f(x) + c \) is \( f(x) \) translated up by \( c \) units.

\( f(x) - c \) is \( f(x) \) translated down by \( c \) units.
\( f(x - c) \) is \( f(x) \) translated right by \( c \) units.

\( f(x + c) \) is \( f(x) \) translated left by \( c \) units.

\(-f(x)\) is \( f(x) \) reflected over the x-axis.

\( f(-x) \) is \( f(x) \) reflected over the y-axis.

\( cf(x) \) is a **vertical stretch/compression**.

**Example:**

\( f(x) \) \hfill 2f(x) \hfill \frac{1}{2}f(x) \hfill f(2-x) \hfill f(\frac{1}{2}x) \hfill f(x)

\( f(c \cdot x) \) is a **horizontal stretch/compression**.

**Example:**

\( f(x) \) \hfill f(2x) \hfill f(\frac{1}{2}x) \hfill f(x) \hfill f(x) \hfill f(x) \)
Here's the order for applying transformations:

1. Horizontal stretch/compression
2. Y-axis reflection
3. Horizontal shift
4. Vertical stretch/compression
5. X-axis reflection

Ex: Sketch the graph of \( y = -\sqrt{x+4} + 2 \)

Solution:

\[
\begin{align*}
y &= \sqrt{x} \\
y &= \sqrt{x+4} \\
&\text{ (left shift)} \\
y &= -\sqrt{x+4} \\
&\text{ (x-axis reflection)} \\
y &= -\sqrt{x+4} + 2 \\
&\text{ (vertical shift)}
\end{align*}
\]