§ 3.3 - Properties of Functions

A function is a rule that assigns to each element of a set exactly one element of another set.

Common notation: $y = f(x)$

- $y =$ dependent variable
- $x =$ independent variable.

Ex: The area of a circle is a function of its radius:

$$A(r) = \pi r^2$$

Ex: The volume of a cube is a function of its side length:

$$V(s) = s^3$$

The domain of a function is the set of all possible input values ($x$ values).

The range of a function is the set of all possible output values ($y$ values).

<table>
<thead>
<tr>
<th>Ex: Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = x^2$</td>
<td>$\mathbb{R}$\ ($(-\infty, \infty)$)</td>
<td>$[0, \infty)$</td>
</tr>
<tr>
<td>$y = \sqrt{x}$</td>
<td>$[0, \infty)$</td>
<td>$[0, \infty)$ (can't output negatives!)</td>
</tr>
<tr>
<td>$y = \cos(x)$</td>
<td>$\mathbb{R}$</td>
<td>$[-1, 1]$</td>
</tr>
</tbody>
</table>
When finding the domain of a function...

1. Don't allow division by 0,
2. Don't $\sqrt{}$ (or even-root) a negative number,
3. Don't take a logarithm of a number $\leq 0$.

Ex: Find the domain of $f(x)$.

1. $f(x) = \frac{1}{x-1}$

Solution: Denominator is 0 when $x=1$, so this point is excluded.

$$\text{Domain} = (-\infty, 1) \cup (1, \infty).$$

2. $f(x) = \sqrt{5-x}$

Solution: If $5-x < 0$, then we can't take $\sqrt{}$.

Note: $5-x \geq 0 \iff x \leq 5$

So, the domain is $[(-\infty, 5]$.

3. $f(x) = \frac{\sqrt{x^2-3x+2}}{x}$

Solution: We can't divide by 0, so $x=0$ is out.

We also need $x^2-3x+2 \geq 0$ to take $\sqrt{}$. 

We factor \( x^2 - 3x + 2 = (x-2)(x-1) \) and check values around the roots:

\[
\begin{array}{cccc}
\text{+} & & \text{-} & \text{+} \\
1 & & 2 & \\
\end{array}
\]

\[ \uparrow \text{Middle interval is bad news!} \]

Combining these restrictions, we get

\[ \text{Domain} = (-\infty, 0) \cup (0, 1] \cup [2, \infty) \]

Q: How do we know if an equation defines a function?

A: We use the vertical line test!

**Vertical Line Test**: If a vertical line passes through more than 1 point of a graph, then this is **NOT** the graph of a function.

Ex:

\[
\begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\]

\[ y = x^2 \text{ IS a function!} \]

\[
\begin{array}{c}
\text{y} \\
\text{x}
\end{array}
\]

\[ x = y^2 \text{ IS NOT a function!} \]

**Evaluating Functions**: To evaluate \( y = f(x) \) at a point \( x = a \), replace every \( x \) in \( f(x) \) with \( a \).
Ex: If \( f(x) = 2x^3 + \sqrt{x} \), then

\[
\begin{align*}
\cdot \ f(1) &= 2(1)^3 + \sqrt{1} = 3 \\
\cdot \ f(4) &= 2(4)^3 + \sqrt{4} = 2 \cdot 64 + 2 = 130 \\
\cdot \ f(x+h) &= 2(x+h)^3 + \sqrt{x+h} \\
\cdot \ f(\heartsuit) &= 2(\heartsuit)^3 + \sqrt{\heartsuit} \\
\text{(It's a hedgehog, okay!?) }
\end{align*}
\]

**Composition of Functions**

We can use functions \( f(x) \) and \( g(x) \) to make new functions!

The composition of \( f \) and \( g \) is the function \( f \circ g \), defined by

\[
(f \circ g)(x) = f(g(x))
\]

(i.e., first apply \( g \), then apply \( f \)).

We can also do the reverse composition \( g \circ f \) (first apply \( g \), then apply \( f \)).

Ex: If \( f(x) = x^2 - x \) and \( g(x) = 2x+1 \), then

\[
\begin{align*}
(f \circ g)(x) &= f(g(x)) = f(2x+1) = (2x+1)^2 - (2x+1) \\
&= (4x^2 + 4x + 1) - (2x+1) \\
&= 4x^2 + 2x \\
\end{align*}
\]

\[
\begin{align*}
(g \circ f)(x) &= g(f(x)) = g(x^2 - x) = 2(x^2 - x) + 1 \\
&= 2x^2 - 2x + 1.
\end{align*}
\]

**Caution:** Usually \( f \circ g \neq g \circ f \)!!