§1.2 - Length, Dot Product, Planes

The length of a vector \( \vec{x} = (x_1, x_2) \) in \( \mathbb{R}^2 \) is defined as
\[
\| \vec{x} \| = \sqrt{x_1^2 + x_2^2}
\]

For \( \vec{x} = (x_1, x_2, x_3) \) in \( \mathbb{R}^3 \), the length is
\[
\| \vec{x} \| = \sqrt{x_1^2 + x_2^2 + x_3^2}
\]

and in general, \( \| (x_1, x_2, \ldots, x_n) \| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \).

The distance between \( \vec{a} \) and \( \vec{b} \) is
\[
\| \vec{a} - \vec{b} \| = \| \vec{b} - \vec{a} \|
\]

Ex: The distance from \( \vec{a} = (1, 1) \) to \( \vec{b} = (2, 4) \) is
\[
\| (2, 4) - (1, 1) \| = \| (1, 3) \| = \sqrt{1^2 + 3^2} = \sqrt{10}
\]

Properties of \( \| \cdot \| \):

1. \( \| \vec{x} \| > 0 \), unless \( \vec{x} = 0 \); \( \| \vec{0} \| = 0 \).
2. \( \| k \vec{x} \| = |k| \| \vec{x} \| \) for \( k \in \mathbb{R} \).
3. \( \| \vec{x} + \vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \| \) for any \( \vec{x}, \vec{y} \) (triangle inequality).
Another operation is the dot product:

\[(a_1, a_2) \cdot (b_1, b_2) = a_1 b_1 + a_2 b_2\]

\[(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3\]

\[\vdots\]

\[(a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n.\]

**Ex:** \[(1, -1, 2) \cdot (3, 4, 5) = (1)(3) + (-1)(4) + (2)(5)\]

\[= 3 - 4 + 10 = 9\]

**Properties of the Dot Product:**

1. \[\vec{a} \cdot \vec{a} = ||a||^2 = a_1^2 + a_2^2 + \ldots + a_n^2\]
   - This will be used a lot!

2. \[\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}\]

3. \[\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}\]

4. \[(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})\text{ for } k \in \mathbb{R}.\]

5. \[\vec{a} \cdot \vec{b} = ||a|| ||b|| \cos \theta\]
   - where \(\theta\) is the angle between \(\vec{a}\) and \(\vec{b}\).

**Note:** \(\vec{a}\) and \(\vec{b}\) are perpendicular (orthogonal) if \(\theta = \frac{\pi}{2}\)

(i.e., if they meet at a 90° angle)
By property 5. of the dot product,

Non-zero \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal \( \iff \mathbf{a} \cdot \mathbf{b} = 0 \).

(Try to prove it!)

**Ex:** Find the angle between \( \mathbf{a} = (1, 1, 0) \) and \( \mathbf{b} = (0, 1, 1) \).

**Solution:** \( \mathbf{a} \cdot \mathbf{b} = \| \mathbf{a} \| \| \mathbf{b} \| \cos \Theta \), so...

\[
(1, 1, 0) \cdot (0, 1, 1) = \| (1, 1, 0) \| \| (0, 1, 1) \| \cos \Theta
\]

\[
\Rightarrow (1)(0) + (1)(1) + (0)(1) = \sqrt{1^2 + 1^2 + 0^2} \sqrt{0^2 + 1^2 + 1^2} \cos \Theta
\]

\[
\Rightarrow 1 = \sqrt{2} \cdot \sqrt{2} \cos \Theta
\]

\[
\Rightarrow \cos \Theta = \frac{1}{2}, \text{ so } \Theta = \frac{\pi}{3}
\]

**Ex:** What value(s) of \( k \) make \( (1, k, 3k) \) orthogonal to \( (1, 2, 0) \)?

**Solution:** They are orthogonal exactly when the dot product is 0.

\[
(1, 2, 0) \cdot (1, k, 3k) = (1)(1) + (2)(k) + (0)(3k)
\]

\[
= 1 + 2k
\]

So \( 1 + 2k = 0 \) \( \Rightarrow 2k = -1 \)

\[
\Rightarrow k = -\frac{1}{2}
\]
Planes in $\mathbb{R}^3$

To define a plane in $\mathbb{R}^3$, we just need to specify:

1. its normal vector $\vec{n}$
   (a vector perpendicular to the plane)

2. a point on the plane.

The scalar equation of a plane is

$$N_1 X_1 + N_2 X_2 + N_3 X_3 = d$$

where $\vec{n}$ is the normal vector and $d$ is found by substituting $(X_1, X_2, X_3)$ with a point on the plane.

Ex: Find the scalar equation of the plane with normal vector $(1, 4, -2)$, and containing $(2, 1, 2)$.

Solution: $\vec{n} = (1, 4, -2)$, so the equation is

$$1X_1 + 4X_2 - 2X_3 = d.$$

Plug in $(X_1, X_2, X_3) = (2, 1, 2)$ to get $d$:

$$1(2) + 4(1) - 2(2) = d \implies d = 2.$$

The equation is $X_1 + 4X_2 - 2X_3 = 2$. 
Two planes are parallel if the normal vector of one plane is a non-zero scalar multiple of the normal vector of the other.

\[ 2x_1 - x_2 + 3x_3 = 5 \quad \text{and} \quad 4x_1 - 2x_2 + 6x_3 = 1 \]

are parallel planes. Why? Their normal vectors are \((2, -1, 3)\) and \((4, -2, 6)\), and \((4, -2, 6) = 2(2, -1, 3)\).

(the values of \(d\) don't matter!)

Two planes are orthogonal (perpendicular) if their normal vectors are orthogonal (dot product = 0).

\[ \text{Ex:} \]

Are the planes \(x_1 + 3x_2 - 4x_3 = 8\) and \(-x_1 + 3x_2 + 2x_3 = 0\) orthogonal?

Normal vectors: \((1, 3, -4), (-1, 3, 2)\)

\((1, 3, -4) \cdot (-1, 3, 2) = -1 + 9 - 8 = 0. \ Yes!\)
Q: Given a line and a plane in $\mathbb{R}^3$, how do we determine the point at which they intersect?

A: Find the parametric equation of the line. Plug these into the plane and solve for $t$.

Ex: Find the intersection of the plane $x_1 + x_2 + 2x_3 = 4$ and the line $(1,2,1) + t(1,2,-1)$, $t \in \mathbb{R}$.

Solution: The parametric equation is

$$x_1 = 1 + t, \quad x_2 = 2 + 2t, \quad x_3 = 1 - t$$

So $x_1 + x_2 + 2x_3 = 4$ $\Rightarrow$ $(1+t) + (2+2t) + 2(1-t) = 4$

$\Rightarrow 5 + t = 4$

$\Rightarrow t = -1$

So $x_1 = 1 + t = 0$
$x_2 = 2 + 2t = 0$
$x_3 = 1 - t = 2$.

They intersect at $(0,0,2)$. 