Dynamics of Continuous, Discrete and Impulsive Systems
Series **A:** *Mathematical Analysis* **11 (2004)** 301-319

OVERLAPPING QUADRATIC OPTIMAL CONTROL OF TIME-VARYING DISCRETE-TIME SYSTEMS

L. Bakule [1] and J. Rodellar [2] and J.M. Rossell [3]

[1]Institute of Information Theory and Automation,
Academy of Sciences of the Czech Republic,182 08 Prague 8, Czech Republic
[2]Department of Applied Mathematics III
Universitat Politècnica de Catalunya (UPC), Campus Nord, C-2, 08034-Barcelona, Spain
[3]Department of Applied Mathematics III
Universitat Politècnica de Catalunya (UPC), Campus Manresa, 08240-Manresa, Spain

Abstract. Overlapping quadratic optimal control of linear time-invariant systems and a commutative class of continuous-time time-varying systems has been recently developed by using a generalized structure of complementary matrices. It has been shown that these structures offer a powerful and effective mean for decentralized control design for these classes of systems. In this paper, these structures are presented for general linear time-varying discrete-time systems. The results presented here concern the transition matrices and explicit conditions on complementary matrices for this class of systems. It essentially differs from continuous-time linear time-varying systems, where general results hold only for aggregations and restrictions. A guideline for their selection is given. The effectiveness of this generalized structure is illustrated by a numerical example of overlapping decentralized control design.
Keywords. Decentralized control, large-scale systems, linear systems, discrete–time systems, time–varying systems, overlapping decompositions, optimal control.
AMS (MOS) subject classification: 93A14, 93A15, 93B17, 93C05, 93C35, 93C50

1 Introduction

Whenever possible, a model simplification in large-scale systems modelling, which results in smaller dimensional models keeping the most important features of the original system, is always a preferable and desirable procedure. One way to follow this line is a mathematical framework for comparing two dynamic systems known as *Inclusion Principle*. It states relations between two dynamic systems with different dimensions under which solutions of the system with larger dimension include solutions of the system with smaller dimension. An original system with common shared parts (*overlapping*) may be expanded into a larger space in order to become disjoint. Thereby, well-established decentralized control methods for disjoint subsystems may be used in the expanded space. Subsequently, such designed controllers are contracted for implementation into the original space.

1.1 Relevant references

The inclusion principle has been developed by Šiljak and his co-workers [6], [7], [12]. The relation between both systems is constructed usually on the base of appropriate linear transformations between the corresponding systems in the original and expanded spaces, where a key role in the selection of appropriate structure of all matrices in the expanded space is played by the so called *complementary matrices* [6], [12]. The conditions on complementary matrices, which are presented there, have implicit character in the sense that it is difficult to select their specific values. In fact only two particular forms of aggregations and restrictions have been used for the control design and numerical computations [12]. Recently, a new generalized structure of complementary matrices has been used as a mean to overcome this drawback [2], [3]. The validity of the general selection of complementary matrices is restricted on the class of continuous-time LTV commutative systems and its extension to discrete-time LTV systems is not at all straightforward [4]. The importance of the inclusion principle is underlined by various applications, for instance in applied mathematics [11], automated highway systems [13], flexible structures [1] or electric power systems [12].

One of the open research issues within the inclusion principle is the extension of the results available for LTI systems to LTV systems. The available results in this direction for continuous-time LTV systems are in [4], [8] and [14]. To the authors knowledge, discrete-time LTV systems are considered only in [5], which serves as a preliminary version of this paper.

1.2 Outline of the paper

The strategy of generalized selection of complementary matrices has been used as an effective tool to find both structure and optimal values of free elements of complementary matrices for LTI systems. These free elements play an important role when considering suboptimality. The main part of this paper concerns an extension of this strategy for overlapping state LQ control for discrete-time LTV systems including the contractibility conditions.

The paper is organized as follows. The problem is formulated in Section 2. The main results are presented in Section 3, identifying a new block structure of the complementary matrices that for the first time present in a compact form results for expansion-contraction of pairs of systems and optimal control criteria. Subsection 3.1 presents general conditions on the complementary matrices in the LQ control of discrete-time LTV systems. It includes aggregations and restrictions as important particular cases. Subsection 3.2 describes expansion-contraction process for the considered problem at a general level by using a new basis and including contractibility conditions. Subsection 3.3 outlines a guideline for selection of these matrices. In Section 4, a numerical example of the overlapping state LQ optimal control design is supplied to illustrate advantages of this procedure.

2 Problem formulation

2.1 Preliminaries

Consider the optimal control problems as follows:

$$\min_{u(k)} J\big(x(k_0), u(k)\big) = x^T(k_f)\Pi\, x(k_f) + \\ + \sum_{k=k_0}^{k_f-1} \left[x^T(k)Q^*(k)x(k) + u^T(k)R^*(k)u(k)\right],$$

$$\text{s.t. } \mathbf{S}: \quad x(k+1) = A(k)x(k) + B(k)u(k) \tag{1}$$

and

$$\min_{\tilde{u}(k)} \tilde{J}\big(\tilde{x}(k_0), \tilde{u}(k)\big) = \tilde{x}^T(k_f)\tilde{\Pi}\, \tilde{x}(k_f) + \\ + \sum_{k=k_0}^{k_f-1} \left[\tilde{x}^T(k)\tilde{Q}^*(k)\tilde{x}(k) + \tilde{u}^T(k)\tilde{R}^*(k)\tilde{u}(k)\right],$$

$$\text{s.t. } \tilde{\mathbf{S}}: \quad \tilde{x}(k+1) = \tilde{A}(k)\tilde{x}(k) + \tilde{B}(k)\tilde{u}(k), \tag{2}$$

where k_0 is the initial time, k_f is the final time and integers $k \in [k_0, k_0 + 1, ..., k_f]$. The vectors $x(k) \in \mathbb{R}^{\mathbf{n}}$, $u(k) \in \mathbb{R}^{\mathbf{m}}$ and $\tilde{x}(k) \in \mathbb{R}^{\tilde{\mathbf{n}}}$, $\tilde{u}(k) \in \mathbb{R}^{\tilde{\mathbf{m}}}$ are the states and inputs of $\mathbf{S}$ and $\tilde{\mathbf{S}}$, respectively, at time k for $k \in [k_0, k_f]$. The matrices $A(k)$, $B(k)$ and $\tilde{A}(k)$, $\tilde{B}(k)$ have $n \times n$, $n \times m$ and $\tilde{n} \times \tilde{n}$, $\tilde{n} \times \tilde{m}$ dimensions, respectively. $Q^*(k)$, $\tilde{Q}^*(k)$ are symmetric, nonnegative definite matrices of dimensions $n \times n$, $\tilde{n} \times \tilde{n}$, respectively. $R^*(k)$, $\tilde{R}^*(k)$ are symmetric, positive definite matrices of dimensions $m \times m$, $\tilde{m} \times \tilde{m}$, respectively. Π, $\tilde{\Pi}$ are constant, symmetric, nonnegative definite matrices of dimensions $n \times n$, $\tilde{n} \times \tilde{n}$, respectively. In problems (1) and (2) the final time k_f is fixed and $x(k_f)$ is free. The minimization of $J(x(k_0), u(k))$ searches for a control $u(k)$ without an excessive effort able to maintain the state vector $x(k)$ close to the zero required state at any time $k \in [k_0, k_f]$, with particular emphasis at the terminal time k_f as weighted by matrix Π. It is well known that the solution of the problem (1) exists, is unique and given in the form $u(k) = -K(k)x(k)$. It is based on the solution of the corresponding Riccati equation [9]. If k_f is finite, this control law ensures a bounded state and the stability issues are absent. If k_f is infinite (with $\Pi = 0$), the question of stability becomes important. There are results ensuring that this control guarantees that the closed-loop system is exponentially stable under certain conditions related to controllability and observability [9]. We assume that the system $\mathbf{S}$ satisfies such conditions. Similar comments hold for problem (2). Suppose that the dimensions of the state and input vectors $x(k)$, $u(k)$ of $\mathbf{S}$ are smaller than (or at most equal to) those of $\tilde{x}(k)$, $\tilde{u}(k)$ of $\tilde{\mathbf{S}}$. Denote $x(k)=x\,(k;x(k_0),u(k))$, $\tilde{x}(k)=\tilde{x}\,(k;\tilde{x}(k_0),\tilde{u}(k))$ the solutions of (1), (2) for given inputs $u(k)$, $\tilde{u}(k)$, respectively. Given an initial time k_0, an initial state $x(k_0)$ and an input signal $u(k)$ defined for all $k\in[k_0,k_f]$, it is well known that

$$\begin{aligned} x(k) &= \Phi(k,k_0)\,x(k_0) + \sum_{j=k_0}^{k-1} \Phi(k,j+1)B(j)u(j), \\ \tilde{x}(k) &= \tilde{\Phi}(k,k_0)\,\tilde{x}(k_0) + \sum_{j=k_0}^{k-1} \tilde{\Phi}(k,j+1)\tilde{B}(j)\tilde{u}(j) \end{aligned} \tag{3}$$

are the unique solutions of the systems (1) and (2), respectively. Φ, $\tilde{\Phi}$ are *discrete-time transition matrices* [10]. Suppose that these sums are zero if $k=k_0$. Denote

$$\Phi(k,j) = A(k-1)A(k-2)\cdots A(j),\; k \geqslant j \tag{4}$$

by adopting the convention that an empty product is the identity, $\Phi(k,j)=I$, if $k=j$.

Let us consider the following linear transformations:

$$\begin{aligned} &V:\mathbb{R}^{\mathbf{n}} \longrightarrow \mathbb{R}^{\tilde{\mathbf{n}}}, \qquad && U:\mathbb{R}^{\tilde{\mathbf{n}}} \longrightarrow \mathbb{R}^{\mathbf{n}}, \\ &R:\mathbb{R}^{\mathbf{m}} \longrightarrow \mathbb{R}^{\tilde{\mathbf{m}}}, && Q:\mathbb{R}^{\tilde{\mathbf{m}}} \longrightarrow \mathbb{R}^{\mathbf{m}} \end{aligned} \tag{5}$$

with rank$(V)=n$, rank$(R)=m$ satisfying $UV=I_n$, $QR=I_m$, where I_n, I_m are identity matrices of indicated dimensions. The systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related through the transformations $\tilde{x}(k)=Vx(k)$, $x(k)=U\tilde{x}(k)$, $\tilde{u}(k)=Ru(k)$, $u(k)=Q\tilde{u}(k)$, where V, U, R and Q are constant matrices of appropriate dimensions and full ranks. Time-varying transformation matrices have been considered in [14].

Definition 1 *A system $\tilde{\mathbf{S}}$ includes the system $\mathbf{S}$, that is $\tilde{\mathbf{S}}\supset\mathbf{S}$, if there exists a quadruplet of constant matrices (U,V,Q,R) such that $UV=I_n$, $QR=I_m$ and for any initial state $x(k_0)$ and any fixed $u(k)$ of $\mathbf{S}$, the choice $\tilde{x}(k_0)=Vx(k_0)$ and $\tilde{u}(k)=Ru(k)$ implies $x(k;x(k_0),u(k))=U\tilde{x}(k;Vx(k_0),Ru(k))$ for all $k\in[k_0,k_f]$.*

Definition 2 *A pair $(\tilde{\mathbf{S}},\tilde{J})$ includes the pair $(\mathbf{S},J)$, that is $(\tilde{\mathbf{S}},\tilde{J})\supset(\mathbf{S},J)$, if $\tilde{\mathbf{S}}\supset\mathbf{S}$ and $J(x(k_0),u(k))=\tilde{J}(Vx(k_0),Ru(k))$ for all $k\in[k_0,k_f]$.*

Definition 3 *If $(\tilde{\mathbf{S}},\tilde{J})\supset(\mathbf{S},J)$, then $(\tilde{\mathbf{S}},\tilde{J})$ is called an expansion of $(\mathbf{S},J)$ and $(\mathbf{S},J)$ is a contraction of $(\tilde{\mathbf{S}},\tilde{J})$.*

There exist two important cases satisfying $\tilde{\mathbf{S}}\supset\mathbf{S}$. They are called *restrictions* and *aggregations*. They are given by the following definitions.

Definition 4 *A system* $\mathbf{S}$ *is a restriction of* $\tilde{\mathbf{S}}$ *if there exists a pair* (V, R) *such that* $UV = I_n$, $QR = I_m$ *for some* U, Q, *and such that for any initial state* $x(k_0)$ *and any fixed input* $u(k)$ *of* $\mathbf{S}$, *the choice* $\tilde{x}(k_0) = Vx(k_0)$ *and* $\tilde{u}(k) = Ru(k)$ *implies* $\tilde{x}(k; \tilde{x}(k_0), \tilde{u}(k)) = Vx(k; x(k_0), u(k))$ *for all* $k \in [k_0, k_f]$.

Definition 5 *A system* $\mathbf{S}$ *is an aggregation of* $\tilde{\mathbf{S}}$ *if there exists a pair* (U, Q) *such that* $UV = I_n$, $QR = I_m$ *for some* V, R, *and such that for any initial state* $\tilde{x}(k_0)$ *and any fixed input* $\tilde{u}(k)$ *of* $\tilde{\mathbf{S}}$, *the choice* $x(k_0) = U\tilde{x}(k_0)$ *and* $u(k) = Q\tilde{u}(k)$ *implies* $x(k; x(k_0), u(k)) = U\tilde{x}(k; \tilde{x}(k_0), \tilde{u}(k))$ *for all* $k \in [k_0, k_f]$.

Definition 6 *A control law* $\tilde{u}(k) = -\tilde{K}(k)\,\tilde{x}(k)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(k) = -K(k)\,x(k)$ *for* $\mathbf{S}$ *if the choice* $\tilde{x}(k_0) = Vx(k_0)$ *and* $\tilde{u}(k) = Ru(k)$ *implies* $K(k)x(k; x(k_0), u(k)) = Q\tilde{K}(k)\tilde{x}(k; Vx(k_0), Ru(k))$ *for all* $k \in [k_0, k_f]$, *any initial state* $x(k_0)$ *and any fixed input* $u(k)$ *of* $\mathbf{S}$.

We may also say that the gain matrix $\tilde{K}(k)$ is contractible to the gain matrix $K(k)$. Contractibility implies that the expanded closed loop system $\tilde{x}(k+1) = [\tilde{A}(k) - \tilde{B}(k)\tilde{K}(k)]\tilde{x}(k)$ includes the closed loop system $x(k+1) = [A(k) - B(k)K(k)]x(k)$.

2.2 The problem

Suppose given the pairs of matrices (U, V) and (Q, R). Then the matrices $\tilde{A}(k)$, $\tilde{B}(k)$, $\tilde{\Pi}$, $\tilde{Q}^*(k)$, $\tilde{R}^*(k)$ and $\tilde{K}(k)$ can be described as

$$\begin{aligned} \tilde{A}(k) &= VA(k)U + M(k), & \tilde{B}(k) &= VB(k)Q + N(k), \\ \tilde{\Pi} &= U^T \Pi U + M_\Pi, & \tilde{Q}^*(k) &= U^T Q^*(k) U + M_{Q^*}(k), \\ \tilde{R}^*(k) &= Q^T R^*(k) Q + N_{R^*}(k), & \tilde{K}(k) &= RK(k)U + F(k) \end{aligned} \tag{6}$$

where $M(k)$, $N(k)$, M_Π, $M_{Q^*}(k)$, $N_{R^*}(k)$ and $F(k)$ are called *complementary matrices*.

The motivation of this work is to obtain a general structure of complementary matrices for overlapping state linear quadratic optimal control problem for discrete-time LTV systems. The specific goals are the following:

- To derive explicit conditions on complementary matrices for discrete-time LTV systems such that $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$, including contractibility conditions.
- To present a guideline for their selections.
- To illustrate the derived results by using a numerical example.

3 Main results

This section gives the results covering mainly the expansion-contraction process for discrete-time LTV systems. Subsection 3.1 includes the general results. Subsection 3.2 characterises the expansion-contraction process presented at a general level by using a new basis and including contractibility conditions. Subsection 3.3 offers a guideline for selection of complementary matrices.

3.1 General discrete-time LTV systems

It is possible to give another definitions, equivalent to Definitions 1 and 6, respectively, in terms of discrete-time transition matrices.

Definition 7 *Consider the systems* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given by (1) and (2), respectively. A system* $\tilde{\mathbf{S}}$ *includes the system* $\mathbf{S}$ *if*

$$U\big[\tilde{\Phi}(k,k_0)\,\tilde{x}(k_0)+\sum_{j=k_0}^{k-1}\tilde{\Phi}(k,j+1)\tilde{B}(j)\tilde{u}(j)\big]= \\ =\Phi(k,k_0)\,x(k_0)+\sum_{j=k_0}^{k-1}\Phi(k,j+1)B(j)u(j) \quad (7)$$

for all $k\in[k_0,k_f]$.

Definition 8 *A control law* $\tilde{u}(k)=-\tilde{K}(k)\,\tilde{x}(k)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(k)=-K(k)\,x(k)$ *for* $\mathbf{S}$ *if*

$$QF(k)\big[\tilde{\Phi}(k,k_0)Vx(k_0)+\sum_{j=k_0}^{k-1}\tilde{\Phi}(k,j+1)\tilde{B}(j)\tilde{u}(j)\big]=0 \quad (8)$$

hold for all fixed $k\in[k_0,k_f]$.

Previously, define for integers $r,s\in[k_0,k_f]$ and any square matrix $\mathcal{M}$ the matrix product $\mathcal{M}[s,r]$ as follows:

$$\begin{array}{l}\mathcal{M}[s,r]=\mathcal{M}(s)\mathcal{M}(s-1)\cdots\mathcal{M}(r),\ \ s>r,\\ \mathcal{M}[s,r]=\mathcal{M}(s),\ \ s=r.\end{array} \quad (9)$$

$\mathcal{M}[s,r]$ is undefined for $s<r$.

Now, we are ready to formulate the Inclusion Principle for discrete-time LTV systems in terms of complementary matrices. Some proofs are omitted because they can be obtained easily.

Theorem 1 *Consider (1) and (2). A pair* $(\tilde{\mathbf{S}},\tilde{J})\supset(\mathbf{S},J)$ *if and only if*

$$\begin{array}{rr}UM[s,r]V=0, & UN(s)R=0,\\ UM[q,p]N(p-1)R=0, & V^TM_\Pi V=0,\\ V^TM_{Q^*}(k)V=0, & R^TN_{R^*}(k)R=0\end{array} \quad (10)$$

hold for all fixed $k\in[k_0,k_f]$, *all* r,s *such that* $k_0\leqslant r\leqslant s\leqslant k-1$ *and all* p,q *such that* $k_0+1\leqslant p\leqslant q\leqslant k-1$.

Proof. The *necessary* and *sufficient* condition are proved simultaneously. Definition 7 is equivalent to the following conditions:
1) $U\tilde{\Phi}(k,k_0)Vx(k_0)=\Phi(k,k_0)x(k_0)$, and
2) $\sum_{j=k_0}^{k-1} U\tilde{\Phi}(k,j+1)\tilde{B}(j)\tilde{u}(j)=\sum_{j=k_0}^{k-1} \Phi(k,j+1)B(j)u(j)$ for all $k\in[k_0,k_f]$.
Substitute $\tilde{\Phi}(k,k_0)=\tilde{A}(k-1)\tilde{A}(k-2)\cdots\tilde{A}(k_0)$ into *1)* where $\tilde{A}(k)=VA(k)U+M(k)$. Substitute $\tilde{\Phi}(k,j+1)=\tilde{A}(k-1)\tilde{A}(k-2)\cdots\tilde{A}(j+1)$ and $\tilde{B}(j)=VB(j)Q+N(j)$ into *2)*. Then, if $k=k_0$, *1)* and *2)* hold directly. From *1)*, for all fixed $k\in[k_0+1,k_f]$ and comparing both sides, we obtain $UM[s,r]V=0$ for all $s\in[k_0,k-1]$ and all $r\in[k_0,s]$, i.e. $UM[s,r]V=0$ for all r,s such that $k_0\leqslant r\leqslant s\leqslant k-1$. By comparing both sides of *2)* for all fixed $k\in[k_0+1,k_f]$, the equation is equivalent to $UN(s)R=0$, $UM[q,p]N(p-1)R=0$, for all $s\in[k_0,k-1]$ and all p,q such that $k_0+1\leqslant p\leqslant q\leqslant k-1$. The remaining conditions are obtained readily when considering $J(x(k_0),u(k))=\tilde{J}(Vx(k_0),Ru(k))$. □

Note. It is important to recognize that it is not necessary to know the transition matrices explicitly in order to select the complementary matrices satisfying the required conditions.

Proposition 1 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given by (1) and (2), respectively. A pair* $(\tilde{\mathbf{S}},\tilde{J})\supset(\mathbf{S},J)$ *if* $V^TM_\Pi V=0$, $V^TM_{Q^*}(k)V=0$, $R^TN_{Q^*}(k)R=0$ *and either*

$$\begin{array}{lll} a) & UM[s,r]=0, \qquad UN(s)=0 & \text{or} \\ b) & M[s,r]V=0, \qquad N(s)R=0 & \end{array} \tag{11}$$

hold for all fixed $k\in[k_0,k_f]$ *and all* r,s *such that* $k_0\leqslant r\leqslant s\leqslant k-1$.

Proof. The proof follows immediately from Theorem 1. □

The conditions *a)* and *b)* are two independent sets of sufficient conditions for $(\tilde{\mathbf{S}},\tilde{J})$ to be an expansion of $(\mathbf{S},J)$.

Proposition 2 *The system* $\mathbf{S}$ *is a restriction of the system* $\tilde{\mathbf{S}}$ *if and only if*

$$\begin{aligned} M[k-1,r]V&=0, & UM[n,m]V&=0, \\ UN(n)R&=0, & N(k-1)R&=0, \\ UM[u,t]N(t-1)R&=0, & M[k-1,p]N(p-1)R&=0 \end{aligned} \tag{12}$$

hold for all fixed $k\in[k_0,k_f]$, *all* r *such that* $k_0\leqslant r\leqslant k-1$, *all* m,n *such that* $k_0\leqslant m\leqslant n\leqslant k-2$, *all* t,u *such that* $k_0+1\leqslant t\leqslant u\leqslant k-2$, *and all* p *such that* $k_0+1\leqslant p\leqslant k-1$.

Proof. Imposing the conditions $\tilde{x}(k;\tilde{x}(k_0),\tilde{u}(k))=Vx(k;x(k_0),u(k))$ given by Definition 4 and following an analogous process as Theorem 1, the proof is concluded. □

Proposition 3 *The system* $\mathbf{S}$ *is an aggregation of the system* $\tilde{\mathbf{S}}$ *if and only if*

$$\begin{aligned} UM[s,k_0]&=0, & UM[q,p]V&=0, \\ UN(s)&=0, & UM[q,p]N(p-1)&=0 \end{aligned} \tag{13}$$

hold for all fixed $k\in[k_0, k_f]$, *all* $s\in[k_0, k-1]$ *and all* p, q *such that* $k_0+1\leqslant p\leqslant q\leqslant k-1$.

Proof. Imposing the conditions $x(k;x(k_0),u(k){=}U\tilde{x}(k;\tilde{x}(k_0),\tilde{u}(k))$ given by Definition 5 and following an analogous process as that one in Theorem 1 the proof is concluded. □

Proposition 4 *The system* $\mathbf{S}$ *is a restriction of the system* $\tilde{\mathbf{S}}$ *if* $M(s)V=0$ *and* $N(s)R=0$ *for all* $s\in[k_0, k-1]$.

Proposition 5 *The system* $\mathbf{S}$ *is an aggregation of the system* $\tilde{\mathbf{S}}$ *if* $UM(s)=0$ *and* $UN(s)=0$ *for all* $s\in[k_0, k-1]$.

The sufficient conditions given by Propositions 4 and 5 in order that $\tilde{\mathbf{S}}\supset\mathbf{S}$ correspond with usual conditions on complementary matrices when considering restrictions and aggregations.

Note. In (12), the conditions $M[k-1,r]V{=}0$ and $UM[n,m]V{=}0$ imply $UM[s,r]V{=}0$ in (10). The relations $UN(n)R{=}0$, $N(k-1)R{=}0$ imply $UN(s)R{=}0$. Finally, $UM[u,t]N(t-1)R{=}0$, $M[k-1,p]N(p-1)R{=}0$ imply $UM[q,p]N(p-1)R{=}0$ for all fixed k and all m,n,p,q,r,s,t,u belonging to the corresponding intervals. Obviously, it is a consequence of the fact that a restriction is a particular case of expansion. A similar note can be make when considering aggregations.

Proposition 6 *A control law* $\tilde{u}(k)=-\tilde{K}(k)\,\tilde{x}(k)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(k)=-K(k)\,x(k)$ *for* $\mathbf{S}$ *if*

$$\begin{array}{cc} QF(k)V=0, & QF(k)M[k-1,r]V=0, \\ QF(k)N(k-1)R=0, & QF(k)M[k-1,p]N(p-1)R=0 \end{array} \tag{14}$$

hold for all fixed $k\in[k_0, k_f]$, *all* $r\in[k_0, k-1]$ *and all* $p\in[k_0+1, k-1]$.

Proof. Consider the following three conditions:
1) $QF(k)\tilde{\Phi}(k,k_0)V{=}0$,
2) $QF(k)\sum_{j=k_0}^{k-1}\tilde{\Phi}(k,j+1)VB(j){=}0$ and
3) $QF(k)\sum_{j=k_0}^{k-1}\tilde{\Phi}(k,j+1)N(j)R{=}0$
for all fixed $k\in[k_0, k_f]$, which imply Definition 8. Substitute $\tilde{\Phi}(k,k_0)=\tilde{A}(k-1)\tilde{A}(k-2)\cdots\tilde{A}(k_0)$ into *1)* and $\tilde{\Phi}(k,j+1){=}\tilde{A}(k-1)\tilde{A}(k-2)\cdots\tilde{A}(j+1)$ into *2)* and *3)* where $\tilde{A}(k){=}VA(k)U+M(k)$. When $k{=}k_0$, only the equation $QF(k_0)V{=}0$ is defined. Suppose that $QF(k)V{=}0$ and $QF(k)M[k-1,r]V{=}0$ hold for all fixed $k\in[k_0+1, k_f]$ and all $r\in[k_0, k-1]$ which imply *1)*. Moreover, *1)* implies *2)*. Suppose that $QF(k)N(k-1)R{=}0$ and $QF(k)M[k-1,p]N(p-1)R{=}0$ for all $p\in[k_0+1, k-1]$ which imply *3)* and the proof is concluded. □

3.2 Expansion–contraction process

Change of basis

The change of basis in the expansion-contraction process introduced in [7], [12] represents $\tilde{\mathbf{S}}$ in a canonical form. Since the inclusion principle does not depend on the specific basis used in the state, input and output spaces, we may introduce convenient changes of basis in $\tilde{\mathbf{S}}$ for a prespecified purpose [2]. The expansion-contraction process between $\mathbf{S}$ and $\tilde{\mathbf{S}}$ can be illustrated in the form

$$\begin{array}{ccccccccc} \mathbf{S} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \tilde{\tilde{\mathbf{S}}} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \mathbf{S}, \\ \mathbb{R}^{\mathbf{n}} & \xrightarrow{V} & \mathbb{R}^{\tilde{\mathbf{n}}} & \xrightarrow{T_A^{-1}} & \mathbb{R}^{\tilde{n}} & \xrightarrow{T_A} & \mathbb{R}^{\tilde{\mathbf{n}}} & \xrightarrow{U} & \mathbb{R}^{\mathbf{n}}, \\ \mathbb{R}^{\mathbf{m}} & \xrightarrow{R} & \mathbb{R}^{\tilde{\mathbf{m}}} & \xrightarrow{T_B^{-1}} & \mathbb{R}^{\tilde{m}} & \xrightarrow{T_B} & \mathbb{R}^{\tilde{\mathbf{m}}} & \xrightarrow{Q} & \mathbb{R}^{\mathbf{m}}, \end{array} \tag{15}$$

where $\tilde{\tilde{\mathbf{S}}}$ denotes the expanded system in the new basis. Given full row-rank matrices V and R, define $U{=}(V^TV)^{-1}V^T$, $Q{=}(R^TR)^{-1}R^T$ as their pseudoinverses, respectively. Let us consider

$$T_A = (V \;\; W_A)\,, \qquad T_B = (R \;\; W_B)\,, \tag{16}$$

where W_A, W_B are chosen such that Im W_A=Ker U, Im W_B=Ker Q. By using these transformations, the conditions $\bar{U}\bar{V}{=}I_n$, $\bar{V}\bar{U}{=}\left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\bar{Q}\bar{R}{=}I_m$, $\bar{R}\bar{Q}{=}\left(\begin{smallmatrix} I_m & 0 \\ 0 & 0 \end{smallmatrix}\right)$ can be easily verified, where $\bar{V}{=}T_A^{-1}V{=}\left(\begin{smallmatrix} I_n \\ 0 \end{smallmatrix}\right)$, $\bar{U}{=}UT_A{=}\left(\begin{smallmatrix} I_n & 0 \end{smallmatrix}\right)$ and $\bar{R}{=}T_B^{-1}R{=}\left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right)$, $\bar{Q} = QT_B{=}\left(\begin{smallmatrix} I_m & 0 \end{smallmatrix}\right)$. Note that the motivating factor for defining T_A and T_B is the fulfillment of these conditions. They play a crucial role in deriving explicit block structured complementary matrices (with zero blocks) including a general strategy for their selection.

Expansion-contraction in the new basis

Consider the system $\mathbf{S}$ partitioned into 3×3 blocks, where the diagonal submatrices $A_{ii}(k)$, $B_{ii}(k)$, i=1, 2, 3 are $n_i \times n_i$, $n_i \times m_i$ matrices, respectively. This structure has been adopted as a prototype structure for overlapping decompositions [7], [8], [12].

Define $(\tilde{\tilde{\mathbf{S}}}, \tilde{\tilde{J}})$ as follows:

$$\min_{\tilde{\tilde{u}}(k)} \; \tilde{\tilde{J}}\big(\tilde{\tilde{x}}(k_0), \tilde{\tilde{u}}(k)\big) = \tilde{\tilde{x}}^T(k_f)\tilde{\tilde{\Pi}}\,\tilde{\tilde{x}}(k_f) + \\ + \sum_{k=k_0}^{k_f-1} \Big[\tilde{\tilde{x}}^T(k)\tilde{\tilde{Q}}^*(k)\tilde{\tilde{x}}(k) + \tilde{\tilde{u}}^T(k)\tilde{\tilde{R}}^*(k)\tilde{\tilde{u}}(k)\Big]\,,$$

$$\text{s.t. } \tilde{\tilde{\mathbf{S}}}: \; \dot{\tilde{\tilde{x}}}(k+1) = \tilde{\tilde{A}}(k)\,\tilde{\tilde{x}}(k) + \tilde{\tilde{B}}(k)\,\tilde{\tilde{u}}(k) \tag{17}$$

for all $k{\in}[k_0, k_f]$, where $\tilde{\tilde{A}}(k)$, $\tilde{\tilde{B}}(k)$, $\tilde{\tilde{\Pi}}$, $\tilde{\tilde{Q}}^*(k)$ and $\tilde{\tilde{R}}^*(k)$ are matrices of appropriate dimensions. The vectors $\tilde{\tilde{x}}(k)$ and $\tilde{\tilde{u}}(k)$ are defined as

$$\tilde{\tilde{x}}(k) = T_A^{-1}Vx(k) = \bar{V}x(k), \qquad \tilde{\tilde{u}}(k) = T_B^{-1}Ru(k) = \bar{R}u(k). \tag{18}$$

Now, the relations between $\tilde{\tilde{\mathbf{S}}}$ and $\mathbf{S}$ are defined as

$$\begin{aligned} \tilde{\tilde{A}}(k) &= \bar{V}A(k)\bar{U} + \bar{M}(k), & \tilde{\tilde{B}}(k) &= \bar{V}B(k)\bar{Q} + \bar{N}(k), \\ \tilde{\tilde{\Pi}} &= \bar{U}^T\Pi\bar{U} + \bar{M}_\Pi, & \tilde{\tilde{Q}}^*(k) &= \bar{U}^TQ^*(k)\bar{U} + \bar{M}_{Q^*}(k), \\ \tilde{\tilde{R}}^*(k) &= \bar{Q}^TR^*(k)\bar{Q} + \bar{N}_{R^*}(k), \end{aligned} \tag{19}$$

where new complementary matrices are

$$\begin{aligned} \bar{M}(k) &= T_A^{-1}M(k)T_A, & \bar{N}(k) &= T_A^{-1}N(k)T_B, \\ \bar{M}_\Pi &= T_A^TM_\Pi T_A, & \bar{M}_{Q^*}(k) &= T_A^TM_{Q^*}(k)T_A, \\ \bar{N}_{R^*}(k) &= T_B^TN_{R^*}(k)T_B. \end{aligned} \tag{20}$$

First, we analyze the structure of the matrices $\bar{M}(k)$, $\bar{N}(k)$, $\bar{M}_\Pi$, $\bar{M}_{Q^*}(k)$ and $\bar{N}_{R^*}(k)$ in the expanded system. Consider the matrices of $\tilde{\mathbf{S}}$ in the form $M(k)=(M_{ij}(k))$, $N(k)=(N_{ij}(k))$, $M_\Pi=(M_{\Pi_{ij}})$, $M_{Q^*}(k)=(M_{Q^*_{ij}}(k))$, $N_{R^*}(k)=(N_{R^*_{ij}}(k))$ for $i,j=1,...,4$, where $M_{\Pi_{ij}}=M^T_{\Pi_{ji}}$, $M_{Q^*_{ij}}(k)=M^T_{Q^*_{ji}}(k)$, $N_{R^*_{ij}}(k)=N^T_{R^*_{ji}}(k)$ and all matrices have corresponding dimensions. Consider $\bar{M}(k)=\begin{pmatrix} \bar{M}_{11}(k) & \bar{M}_{12}(k) \\ \bar{M}_{21}(k) & \bar{M}_{22}(k) \end{pmatrix}$, $\bar{N}(k)=\begin{pmatrix} \bar{N}_{11}(k) & \bar{N}_{12}(k) \\ \bar{N}_{21}(k) & \bar{N}_{22}(k) \end{pmatrix}$, $\bar{M}_\Pi=\begin{pmatrix} \bar{M}_{\Pi_{11}} & \bar{M}_{\Pi_{12}} \\ \bar{M}^T_{\Pi_{12}} & \bar{M}_{\Pi_{22}} \end{pmatrix}$, $\bar{M}_{Q^*}(k)=\begin{pmatrix} \bar{M}_{Q^*_{11}}(k) & \bar{M}_{Q^*_{12}}(k) \\ \bar{M}^T_{Q^*_{12}}(k) & \bar{M}_{Q^*_{22}}(k) \end{pmatrix}$, $\bar{N}_{R^*}(k)=\begin{pmatrix} \bar{N}_{R^*_{11}}(k) & \bar{N}_{R^*_{12}}(k) \\ \bar{N}^T_{R^*_{12}}(k) & \bar{N}_{R^*_{22}}(k) \end{pmatrix}$ where $\bar{M}_{11}(k)$, $\bar{M}_{22}(k)$ are $n\times n$, $n_2\times n_2$ matrices, respectively. $\bar{N}_{11}(k)$, $\bar{N}_{22}(k)$ are $n\times m$, $n_2\times m_2$ matrices, respectively. $\bar{M}_{\Pi_{11}}$, $\bar{M}_{\Pi_{22}}$ are $n\times n$, $n_2\times n_2$ matrices, respectively. $\bar{M}_{Q^*_{11}}(k)$, $\bar{M}_{Q^*_{22}}(k)$ are $n\times n$, $n_2\times n_2$ matrices, respectively. $\bar{N}_{R^*_{11}}(k)$, $\bar{N}_{R^*_{22}}(k)$ are $m\times m$, $m_2\times m_2$ matrices, respectively. We need to know the conditions that these submatrices must verify.

Theorem 2 *A system $\tilde{\tilde{\mathbf{S}}}$ includes the system $\mathbf{S}$ if and only if $\bar{M}(s)$ and $\bar{N}(s)$ have the structure $\bar{M}(s) = \begin{pmatrix} 0 & \bar{M}_{12}(s) \\ \bar{M}_{21}(s) & \bar{M}_{22}(s) \end{pmatrix}$, $\bar{N}(s) = \begin{pmatrix} 0 & \bar{N}_{12}(s) \\ \bar{N}_{21}(s) & \bar{N}_{22}(s) \end{pmatrix}$ and*

$$\begin{aligned} \bar{M}_{12}(p)\bar{M}_{21}(p-1) &= 0, & \bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{M}_{21}(j-1) &= 0, \\ \bar{M}_{12}(p)\bar{N}_{21}(p-1) &= 0, & \bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{N}_{21}(j-1) &= 0 \end{aligned} \tag{21}$$

hold for all fixed $k\in[k_0,k_f]$, all $s\in[k_0,k-1]$, all $p\in[k_0+1,k-1]$ and all $j\in[k_0+1,p-1]$.

Proof. Denote $\bar{M}(k)=\begin{pmatrix} \bar{M}_{11}(k) & \bar{M}_{12}(k) \\ \bar{M}_{21}(k) & \bar{M}_{22}(k) \end{pmatrix}$. Consider $\bar{U}\bar{M}[s,r]\bar{V}=0$, for $k_0\leqslant r\leqslant s\leqslant k-1$ given by Theorem 1. $\bar{U}\bar{M}[k_0,k_0]\bar{V}=0$ implies $\bar{M}_{11}(k_0)=0$ for $k=k_0+1$. We obtain $\bar{M}_{11}(k_0)=0$, $\bar{M}_{11}(k_0+1)=0$ and $\bar{M}_{12}(k_0+1)\bar{M}_{21}(k_0)=0$ for $k=k_0+2$. We get $\bar{M}_{11}(s)=0$ together with the conditions $\bar{M}_{12}(p)\bar{M}_{21}(p-1)=0$, $\bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{M}_{21}(j-1)=0$ for all $s\in[k_0,k-1]$, all $p\in[k_0+1,k-1]$ and all $j\in[k_0+1,p-1]$ when repeating this process. Similarly, the conditions $\bar{M}_{12}(p)\bar{N}_{21}(p-1)=0$, $\bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{N}_{21}(j-1)=0$ are obtained by using the relations $\bar{U}\bar{N}(s)\bar{R}=0$, $\bar{U}\bar{M}[q,p]\bar{N}(p-1)\bar{R}=0$, respectively, given by Theorem 1. □

Proposition 7 *Consider* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given in (1) and (17), respectively. A pair* $(\tilde{\tilde{\mathbf{S}}},\tilde{\tilde{J}})\supset(\mathbf{S},J)$ *if* $\bar{M}_{\Pi}=\begin{pmatrix} 0 & \bar{M}_{\Pi_{12}} \\ \bar{M}_{\Pi_{12}}^T & \bar{M}_{\Pi_{22}} \end{pmatrix}$, $\bar{M}_{Q^*}(k)=\begin{pmatrix} 0 & \bar{M}_{Q^*_{12}}(k) \\ \bar{M}^T_{Q^*_{12}}(k) & \bar{M}_{Q^*_{22}}(k) \end{pmatrix}$, $\bar{N}_{R^*}(k)=\begin{pmatrix} 0 & \bar{N}_{R^*_{12}}(k) \\ \bar{N}^T_{R^*_{12}}(k) & \bar{N}_{R^*_{22}}(k) \end{pmatrix}$ *and either*

$$
\begin{array}{ll}
a) & \bar{M}(p)=\begin{pmatrix} 0 & 0 \\ \bar{M}_{21}(p) & \bar{M}_{22}(p) \end{pmatrix}, \quad \bar{N}(p)=\begin{pmatrix} 0 & 0 \\ \bar{N}_{21}(p) & \bar{N}_{22}(p) \end{pmatrix} \\
or & \\
b) & \bar{M}(p)=\begin{pmatrix} 0 & \bar{M}_{12}(p) \\ 0 & \bar{M}_{22}(p) \end{pmatrix}, \quad \bar{N}(p)=\begin{pmatrix} 0 & \bar{N}_{12}(p) \\ 0 & \bar{N}_{22}(p) \end{pmatrix}
\end{array}
\tag{22}
$$

hold for all fixed $k\in[k_0,k_f]$ *and all* $p\in[k_0+1,k-1]$.

Proof. Considering the conditions *a)* and *b)* given by Proposition 1, respectively, in the new expanded system $\tilde{\tilde{\mathbf{S}}}$, the proof is concluded. □

Contractibility

The idea is to design a control law for $\tilde{\mathbf{S}}$ so that it can be contracted and implemented into $\mathbf{S}$. Now, we want to determine the conditions under which a control law designed in $\tilde{\tilde{\mathbf{S}}}$ can be contracted into $\mathbf{S}$ in terms of complementary matrices.

Denote matrices appearing in the contractibility process as follows. The complementary matrix $F(k)$ has the form $F(k)=(F_{ij}(k))$, $i,j=1,...,4$, where $F_{11}(k)$, $F_{22}(k)$, $F_{33}(k)$ and $F_{44}(k)$ are $m_1\times n_1$, $m_2\times n_2$, $m_2\times n_2$ and $m_3\times n_3$ matrices, respectively. Define $\bar{F}(k)=\begin{pmatrix} \bar{F}_{11}(k) & \bar{F}_{12}(k) \\ \bar{F}_{21}(k) & \bar{F}_{22}(k) \end{pmatrix}$, where $\bar{F}_{11}(k)$ and $\bar{F}_{22}(k)$ are $m\times n$ and $m_2\times n_2$ matrices, respectively. Similarly, denote the gain matrix $K(k)=(K_{ij}(k))$, $i,j=1,2,3$, where $K_{ii}(k)$ are $m_i\times n_i$ matrices, respectively. The gain matrix $\tilde{\tilde{K}}(k)$ for $\tilde{\tilde{\mathbf{S}}}$ has the form $\tilde{\tilde{K}}(k)=\bar{R}K(k)\bar{U}+\bar{F}(k)$, where $\tilde{\tilde{K}}(k)=T_B^{-1}\tilde{K}(k)T_A$ and

$$
\bar{F}(k)=T_B^{-1}F(k)T_A. \tag{23}
$$

So far we do not know the form of the complementary matrix $F(k)$ and the corresponding contractibility conditions. The following theorem solves the problem.

Proposition 8 *A control law* $\tilde{\tilde{u}}(k)=-\tilde{\tilde{K}}(k)\tilde{\tilde{x}}(k)$ *for* $\tilde{\tilde{\mathbf{S}}}$ *is contractible to the control law* $u(k)=-K(k)x(k)$ *of* $\mathbf{S}$ *if* $\bar{F}(k)=\begin{pmatrix} 0 & \bar{F}_{12}(k) \\ \bar{F}_{21}(k) & \bar{F}_{22}(k) \end{pmatrix}$ *and*

$$
\begin{array}{cc}
\bar{F}_{12}(k)\bar{M}_{21}(k-1)=0, & \bar{F}_{12}(k)\bar{N}_{21}(k-1)=0, \\
\bar{F}_{12}(k)\bar{M}_{22}[k-1,p]\bar{M}_{21}(p-1)=0, & \bar{F}_{12}(k)\bar{M}_{22}[k-1,p]\bar{N}_{21}(p-1)=0
\end{array}
\tag{24}
$$

hold for all fixed $k\in[k_0,k_f]$ *and all* $p\in[k_0+1,k-1]$.

Proof. Suppose $\bar{F}(k)=\left(\begin{smallmatrix}\bar{F}_{11}(k) & \bar{F}_{12}(k)\\ \bar{F}_{21}(k) & \bar{F}_{22}(k)\end{smallmatrix}\right)$. Consider the conditions by Proposition 6. Only the relation $\bar{Q}\bar{F}(k)\bar{V}=0$ is defined for $k=k_0$, and then $\bar{F}_{11}(k_0)=0$ is obtained. From $\bar{Q}\bar{F}(k)\bar{V}=0$, $\bar{Q}\bar{F}(k)\bar{M}[k-1,r]\bar{V}=0$ and $\bar{Q}\bar{F}(k)\bar{N}(k-1)\bar{R}=0$ we get $\bar{F}_{11}(k_0+1)=0$, $\bar{F}_{12}(k_0+1)\bar{M}_{21}(k_0)=0$ and $\bar{F}_{12}(k_0+1)\bar{N}_{21}(k_0)=0$ for $k=k_0+1$, respectively. Repeating this process for all k, $\bar{F}_{11}(k)=0$, $\bar{F}_{12}(k)\bar{M}_{21}(k-1)=0$, $\bar{F}_{12}(k)\bar{N}_{21}(k-1)=0$, $\bar{F}_{12}(k)\bar{M}_{22}[k-1,p]\bar{M}_{21}(p-1)=0$ and $\bar{F}_{12}(k)\bar{M}_{22}[k-1,p]\bar{N}_{21}(p-1)=0$ for all fixed $k\in[k_0,k_f]$ and all $p\in[k_0+1,k-1]$ are obtained and the proof is concluded. □

3.3 Guideline for selection of complementary matrices

The above results give a formal block structure of the complementary matrices $\bar{M}(k)$, $\bar{N}(k)$, $\bar{M}_{\Pi}$, $\bar{M}_{Q^*}(k)$, $\bar{N}_{R^*}(k)$ and $\bar{F}(k)$ in the new basis. They are general, in the sense that the obtained structure does not depend on any particular selection of the transformation matrices V and R. Therefore, they hold for any expansion-contraction process.

However, an important practical question arises when the goal is to select specific numerical values of the complementary matrices for a given problem. In this case, it is necessary to translate the structure of the complementary matrices $\bar{M}(k),\ldots,\bar{N}_{R^*}(k)$ and $\bar{F}(k)$ into the corresponding structure of the complementary matrices $M(k),\ldots,N_{R^*}(k)$ and $F(k)$ in the original basis. To do this, specific matrices V and R have to be selected to expand the given problem, such that the changes of basis T_A and T_B in (16) are defined, which are needed to relate the matrices $M(k),\ldots,N_{R^*}(k)$ and $F(k)$ with those $\bar{M}(k),\ldots,\bar{N}_{R^*}(k)$ and $\bar{F}(k)$ as given in (20) and (23). The choice of the expansion matrices is limited by the information structure constraints requiring the preservation of the integrity of the local feedback and subsystems in overlapping decentralized control. Although the resulting structure of the complementary matrices $M(k),\ldots,N_{R^*}(k)$ and $F(k)$ is then problem dependent, it is sufficiently general to give a rich margin for selecting numerical values. For this selection, the designer can follow different specifications. It can be advisable to simplify the structure as much as possible within given constraints by fixing a possible maximum of zero blocks and leaving nonzero blocks with some free parameters for optimization. The authors believe that it is not possible to give an explicit numerical algorithm for a general scenario. The above recommendations can serve as a guideline to build a selection procedure for any given problem.

In the remaining of this section, the structure of the complementary matrices in the initial basis is derived for a specific expansion scheme. Further, the next section gives a numerical example.

We select the following expansion transformation matrices, which are the most frequently used in the literature [12]:

$$V = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix}, \quad R = \begin{pmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix}. \tag{25}$$

By using these transformations, $x_2(k)$ and $u_2(k)$ appear in a repeated form in $\tilde{x}(k)=\big(x_1^T(k), x_2^T(k), x_2^T(k), x_3^T(k)\big)^T$ and $\tilde{u}(k)=\big(u_1^T(k), u_2^T(k), u_2^T(k), u_3^T(k)\big)^T$ by using (25), respectively. The change of basis results in

$$T_{_A} = \begin{pmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & I_{n_2} \\ 0 & I_{n_2} & 0 & -I_{n_2} \\ 0 & 0 & I_{n_3} & 0 \end{pmatrix} \tag{26}$$

and its corresponding inverse $T_{_A}^{-1}$. The matrices $T_{_B}$, $T_{_B}^{-1}$ have an analogous structure. The following theorems present the structure of the complementary matrices $M(k)$, $N(k)$, M_Π, $M_{Q^*}(k)$, $N_{R^*}(k)$ and $F(k)$ in the initial basis.

Theorem 3 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given by (1) and (2), respectively. A system* $\tilde{\mathbf{S}}$ *includes the system* $\mathbf{S}$ *if and only if* $M(s)$ *has the following structure*

$$M(s) = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22}+M_{23}+M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix}(s) \tag{27}$$

and their blocks satisfy

$$\begin{aligned}
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(p-1) = 0, \\
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}\big[p-1, j\big]\begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(j-1) = 0, \\
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(p-1) = 0, \\
&\begin{pmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{pmatrix}(p)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}\big[p-1, j\big]\begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(j-1) = 0
\end{aligned} \tag{28}$$

for all fixed $k\in[k_{_0}, k_{_f}]$, *all* $s\in[k_{_0}, k-1]$, *all* $p\in[k_{_0}+1, k-1]$ *and all* $j\in[k_{_0}+1, p-1]$, *where the matrix* $N(s)$ *has the same structure as* $M(s)$.

Proof. Consider $\bar{M}(s)=T_{_A}^{-1}M(s)T_{_A}$. From Theorem 2, $\bar{M}_{11}(s)=0$ and the remaining matrix blocks $\bar{M}_{ij}(s)$, $i,j=1,2$, can be identified and $M(s)$ is obtained. An analogous procedure holds for the matrix $N(s)$ when applying Theorem 2. Now, (28) is obtained imposing $\bar{M}_{12}(p)\bar{M}_{21}(p-1)=0$, $\bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{M}_{21}(j-1)=0$, $\bar{M}_{12}(p)\bar{N}_{21}(p-1)=0$ and $\bar{M}_{12}(p)\bar{M}_{22}[p-1,j]\bar{N}_{21}(j-1)=0$ for all fixed $k\in[k_{_0}, k_{_f}]$, all $p\in[k_{_0}+1, k-1]$ and all $j\in[k_{_0}+1, p-1]$ given by Theorem 2. □

We may identify two important cases from (28):

$$\begin{array}{llll} a) & M_{12}(p)=0, & \big(M_{23}+M_{33}\big)(p)=0, & M_{42}(p)=0, \\ b) & M_{21}(p-1)=0, & \big(M_{22}+M_{23}\big)(p-1)=0, & M_{24}(p-1)=0, \\ & N_{21}(p-1)=0, & \big(N_{22}+N_{23}\big)(p-1)=0, & N_{24}(p-1)=0 \end{array} \qquad (29)$$

for all fixed $k\in[k_0,k_f]$ and all $p\in[k_0+1,k-1]$. The above case b) corresponds with restrictions.

Proposition 9 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given by (1) and (2). A pair* $(\tilde{\mathbf{S}},\tilde{J})\supset(\mathbf{S},J)$ *if the matrices* M_Π, $M_{Q^*}(k)$, $N_{R^*}(k)$ *have the structure*

$$M_\Pi=\begin{pmatrix} 0 & M_{\Pi_{12}} & -M_{\Pi_{12}} & 0 \\ M_{\Pi_{12}}^T & -M_{\Pi_{23}}-M_{\Pi_{23}}^T-M_{\Pi_{33}} & M_{\Pi_{23}} & M_{\Pi_{24}} \\ -M_{\Pi_{12}}^T & M_{\Pi_{23}}^T & M_{\Pi_{33}} & -M_{\Pi_{24}} \\ 0 & M_{\Pi_{24}}^T & -M_{\Pi_{24}}^T & 0 \end{pmatrix},$$

$$M_{Q^*}(k)=\begin{pmatrix} 0 & M_{Q^*_{12}} & -M_{Q^*_{12}} & 0 \\ M_{Q^*_{12}}^T & -M_{Q^*_{23}}-M_{Q^*_{23}}^T-M_{Q^*_{33}} & M_{Q^*_{23}} & M_{Q^*_{24}} \\ -M_{Q^*_{12}}^T & M_{Q^*_{23}}^T & M_{Q^*_{33}} & -M_{Q^*_{24}} \\ 0 & M_{Q^*_{24}}^T & -M_{Q^*_{24}}^T & 0 \end{pmatrix}(k), \qquad (30)$$

$$N_{R^*}(k)=\begin{pmatrix} 0 & N_{R^*_{12}} & -N_{R^*_{12}} & 0 \\ N_{R^*_{12}}^T & -N_{R^*_{23}}-N_{R^*_{23}}^T-N_{R^*_{33}} & N_{R^*_{23}} & N_{R^*_{24}} \\ -N_{R^*_{12}}^T & N_{R^*_{23}}^T & N_{R^*_{33}} & -N_{R^*_{24}} \\ 0 & N_{R^*_{24}}^T & -N_{R^*_{24}}^T & 0 \end{pmatrix}(k),$$

and either

$$a)\quad M(p)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -M_{22} & -M_{23} & -M_{24} \\ 0 & 0 & 0 & 0 \end{pmatrix}(p),$$

$$N(p)=\begin{pmatrix} 0 & N_{12} & -N_{12} & 0 \\ N_{21} & N_{22} & N_{23} & N_{24} \\ -N_{21} & -(N_{22}+N_{23}+N_{33}) & N_{33} & -N_{24} \\ 0 & N_{42} & -N_{42} & 0 \end{pmatrix}(p) \qquad (31)$$

or

$$b)\quad M(p)=\begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ 0 & M_{22} & -M_{22} & 0 \\ 0 & M_{32} & -M_{32} & 0 \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix}(p),\quad N(p)=\begin{pmatrix} 0 & N_{12} & -N_{12} & 0 \\ 0 & N_{22} & -N_{22} & 0 \\ 0 & N_{32} & -N_{32} & 0 \\ 0 & N_{42} & -N_{42} & 0 \end{pmatrix}(p)$$

hold for all fixed $k\in[k_0,k_f]$ *and all* $p\in[k_0+1,k-1]$.

Proof. The proof of the assertion is straightforward when applying Proposition 7. □

Proposition 10 *A control law* $\tilde{u}(k)=-\tilde{K}(k)\tilde{x}(k)$ *for* $\tilde{\mathbf{S}}$ *is contractible to* $u(k)=-K(k)x(k)$ *of* $\mathbf{S}$ *if* $F(k)=\begin{pmatrix} 0 & F_{12} & -F_{12} & 0 \\ F_{21} & F_{22} & F_{23} & F_{24} \\ -F_{21} & -(F_{22}+F_{23}+F_{33}) & F_{33} & -F_{24} \\ 0 & F_{42} & -F_{42} & 0 \end{pmatrix}(k)$ *and* *the equations*

$$
\begin{aligned}
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(k-1)=0,\\
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(k-1)=0,\\
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}\big[k-1,p\big]\begin{pmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{pmatrix}(p-1)=0,\\
&\begin{pmatrix} F_{12} \\ F_{23}+F_{33} \\ F_{42} \end{pmatrix}(k)\begin{pmatrix} M_{22}+M_{33} \end{pmatrix}\big[k-1,p\big]\begin{pmatrix} N_{21} & N_{22}+N_{23} & N_{24} \end{pmatrix}(p-1)=0
\end{aligned}
\tag{32}
$$

hold for all fixed $k\in[k_0, k_f]$ *and all* $p\in[k_0+1, k-1]$.

Proof. Consider $\bar{F}(k)=T_B^{-1}F(k)T_A$ with T_B^{-1}, T_A given by (26). $\bar{F}_{11}(k)=0$ and the other matrix blocks $\bar{F}_{ij}(k)$, $i,j=1,2$ can be identified from Proposition 8. Thus, the structure of $F(k)$ is obtained. Further, we get (32) imposing (24). $\square$

From Proposition 10 by choosing $F_{12}(k)=0$, $\big(F_{23}+F_{33}\big)(k)=0$ and $F_{42}(k)=0$ for all $k\in[k_0, k_f]$, any control law designed for $\tilde{\mathbf{S}}$ is contractible to $\mathbf{S}$.

4 An example

Objective

Consider the problem (1) with the specific matrices

$$
A(k)=\begin{pmatrix} 1 & 0 & 0 & -1 \\ k & -1 & -1 & 0 \\ 0 & 0 & 0 & k \\ 1 & 0 & -2 & 0 \end{pmatrix},\quad B(k)=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & k \\ 1 & -1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \tag{33}
$$

$\Pi=Q^*=\mathrm{diag}(1,1,1,1)$, $R^*=\mathrm{diag}(1,1,1)$ and initial state $x(k_0)=(1,1,1,1)^T$. The overlapping decomposition is determined by dashed lines. Consider the initial and the final time as $k_0=0$ and $k_f=3$, respectively. It is well known that a discrete-time time-varying system is controllable if and only if the matrix $\sum_{k=k_0}^{k_f}\Phi(k_f,k)B(k)B^T(k)\Phi(k_f,k)$ is invertible [9]. The system (33) is controllable.

The objective is to show the potential advantages that offer the characterization of the presented complementary matrices for an overlapping decentralized state LQ optimal control design.

We consider the following steps:

1) The pair $(\mathbf{S}, J)$ in (1) is expanded to $(\tilde{\mathbf{S}}, \tilde{J})$ by using the transformations V and R given in (25). The system $\tilde{\mathbf{S}}$ can be represented as:

$$\tilde{\mathbf{S}}: \ \tilde{x}(k+1) = \tilde{A}_D(k)\,\tilde{x}(k) + \tilde{B}_D(k)\,\tilde{u}(k) + \tilde{A}_C(k)\,\tilde{x}(k) + \tilde{B}_C(k)\,\tilde{u}(k) \tag{34}$$

with $\tilde{A}_D(k)$, $\tilde{B}_D(k)$ block diagonal matrices and $\tilde{A}_C(k)$, $\tilde{B}_C(k)$ the corresponding interconnection matrices. Check the controllability of the pair $(\tilde{A}_D(k), \tilde{B}_D(k))$.

2) A decentralized control law $\tilde{u}_D(k) = -\tilde{K}_D(k)\,\tilde{x}(k)$ is designed for the decoupled expanded system $\tilde{\mathbf{S}}_{\mathbf{D}}: \ \tilde{x}(k+1) = \tilde{A}_D(k)\,\tilde{x}(k) + \tilde{B}_D(k)\,\tilde{u}(k)$, where $\tilde{K}_D(k)$ satisfies the equation

$$\tilde{K}_D(k) = \left\{\tilde{R}^* + \tilde{B}_D^T(k)[\tilde{Q}^* + \tilde{P}_D(k+1)]\tilde{B}_D(k)\right\}^{-1}\tilde{B}_D^T(k)[\tilde{Q}^* + \tilde{P}_D(k+1)]\tilde{A}_D(k) \tag{35}$$

together with

$$\tilde{P}_D(k) = \tilde{A}_D^T(k)[\tilde{Q}^* + \tilde{P}_D(k+1)][\tilde{A}_D(k) - \tilde{B}_D(k)\tilde{K}_D(k)] \tag{36}$$

for all $k = k_0, ..., k_f - 1$, with the boundary condition $\tilde{P}_D(k_f) = \tilde{\Pi}_D$. Select $F(k)$ satisfying the contractibility conditions given by Proposition 10.

3) Contract $u_D(k) = -Q\tilde{K}_D(k)Vx(k) = -K_D(k)x(k)$ to be implemented into $\mathbf{S}$. The value of the performance index (1) obtained by $u_D(k)$ is not optimal, in general. We can compare $J^\circ(x(k_0))$ of the centralized optimal control law that minimizes (1) with the value obtained by the implementation of the control $u_D(k)$ into $\mathbf{S}$, which is denoted by $J^\oplus(x(k_0))$.

In this example we consider the matrices $N(k)=0$, $M_\Pi=0$, $M_{Q^*}(k)=0$, $N_{R^*}(k)=0$ and $F(k)=0$, which are particular simple cases satisfying Propositions 9 and 10. Consequently, because $F(k)=0$, the contraction of the control law designed in the expanded space is possible. The complementary matrix $M(k)$ is selected within the cases described in the previous section and $J^\oplus(x(k_0))$ is computed. To evaluate and compare the results obtained by the proposed method we will also consider the selection of $M(k)$ corresponding to the case of restrictions.

Results

Overlapping decomposition by using a restriction. Choose the typical complementary matrix $M(k)$ used in [8], [12]. Then, $M(k)$ and $\tilde{A}(k)$ are given by

$$M(k) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & -0.5 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & -0.5 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}, \quad \tilde{A}(k) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ k & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k \\ k & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & k \\ 1 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}. \tag{37}$$

The decoupled matrices $\tilde{A}_{D}(k)$ and $\tilde{B}_{D}(k)$ are in this situation given by

$$\tilde{A}_{D}(k)=\begin{pmatrix}1&0&0&0&0&0\\k&-1&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&-1&-1&0\\0&0&0&0&0&k\\0&0&0&0&-2&0\end{pmatrix},\quad \tilde{B}_{D}(k)=\begin{pmatrix}1&0&0&0\\0&0&0&0\\1&-0.5&0&0\\0&0&0&k\\0&0&-0.5&0\\0&0&-0.5&0\end{pmatrix}. \tag{38}$$

The contracted gain matrices, $K_{D}(k)$, $k{=}0,1,2$, computed from step 3), are the following:

$$\begin{aligned}K_{D}(0)&=\begin{pmatrix}0.56&-0.09&-0.09&0.00\\0.28&0.28&0.87&0.00\\0.00&0.00&0.00&0.00\end{pmatrix},\quad K_{D}(1)=\begin{pmatrix}0.64&0.20&0.20&0.00\\0.19&0.26&1.04&-0.47\\0.00&-0.41&-0.32&0.06\end{pmatrix},\\K_{D}(2)&=\begin{pmatrix}0.54&0.00&0.00&0.00\\0.18&0.00&0.80&-0.40\\0.00&-0.40&-0.40&0.00\end{pmatrix}.\end{aligned}\tag{39}$$

Overlapping decomposition by using the proposed method. Suppose the selection of the transformation matrices according to the guideline in Subsection 3.3. Then, the zero blocks of $M(k)$ have been selected in such a way that $M_{12}(p){=}0$, $\big(M_{23}+M_{33}\big)(p){=}0$, $\big(M_{22}+M_{23}\big)(p-1){=}0$, $M_{24}(p-1){=}0$ and $\big(M_{22}+M_{33}\big)[p-1,j]{=}0$ for all fixed $k{\in}[k_0,k_f]$, all $s{\in}[k_0,k-1]$, all $p{\in}[k_0+1,k-1]$ and all $j{\in}[k_0+1,p-1]$. The remaining nonzero blocks have been considered with a fixed submatrix $M_{42}(p){=}\begin{pmatrix}0&1\end{pmatrix}$ and $M_{21}(p-1){=}\begin{pmatrix}-k\\0\end{pmatrix}$ with a free parameter k for optimization. It satisfies the necessary constraint $M_{42}(p)M_{21}(p-1){=}0$. Such particular selection satisfies the constraints (28). $M(k)$ and $\tilde{A}(k)$ are given by

$$M(k)=\begin{pmatrix}0&0&0&0&0&0\\-k&0&0&0&0&0\\0&0&0&0&0&0\\k&0&0&0&0&0\\0&0&0&0&0&0\\0&0&1&0&-1&0\end{pmatrix},\quad \tilde{A}(k)=\begin{pmatrix}1&0&0&0&0&-1\\0&-0.5&-0.5&-0.5&-0.5&0\\0&0&0&0&0&k\\2k&-0.5&-0.5&-0.5&-0.5&0\\0&0&0&0&0&k\\1&0&0&0&-2&0\end{pmatrix}. \tag{40}$$

The decoupled matrices $\tilde{A}_{D}(k)$ and $\tilde{B}_{D}(k)$ are

$$\tilde{A}_{D}(k)=\begin{pmatrix}1&0&0&0&0&0\\0&-0.5&-0.5&0&0&0\\0&0&0&0&0&0\\0&0&0&-0.5&-0.5&0\\0&0&0&0&0&k\\0&0&0&0&-2&0\end{pmatrix},\quad \tilde{B}_{D}(k)=\begin{pmatrix}1&0&0&0\\0&0&0&0\\1&-0.5&0&0\\0&0&0&k\\0&0&-0.5&0\\0&0&-0.5&0\end{pmatrix}. \tag{41}$$

The matrix $M(k)$ satisfies the structural requirements given by Theorem 3. The corresponding contracted gain matrices $K_{D}(k)$ are given by

$$\begin{aligned}K_{D}(0)&=\begin{pmatrix}0.55&-0.03&-0.03&0.00\\0.13&0.03&0.63&0.00\\0.00&0.00&0.00&0.00\end{pmatrix},\quad K_{D}(1)=\begin{pmatrix}0.55&-0.03&-0.03&0.00\\0.15&0.04&0.83&-0.47\\0.00&-0.18&-0.15&0.02\end{pmatrix},\\K_{D}(2)&=\begin{pmatrix}0.55&0.00&0.00&0.00\\0.18&0.00&0.80&-0.40\\0.00&-0.20&-0.20&0.00\end{pmatrix}.\end{aligned}\tag{42}$$

Following the steps given above, we can summarize the obtained results as follows:

proposed method	restriction	centralized case
$J^{\oplus}=12.8$	$J^{\oplus}=38.3$	$J^{\circ}=6.7$

Note that J° is the cost for the centralized optimal control solving (1). Since the goal of a decentralized control is to drive the system as close as possible to the (ideal) centralized control, we may observe essentially better performance when using the proposed method for the selection of the complementary matrices. Thereby, we have reduced the cost approximately to one third compared with the cost in the case of restriction. This method illustrates the freedom introduced by this approach to select the convenient structure of complementary matrices to reduce the cost in overlapping quadratic optimal control.

By comparing (33) and (37) (restriction), it can be seen that the expanded system preserves the identity of the initial overlapping structure, as the original overlapping subsystems can be recognized within the matrix $\tilde{A}(k)$. This identity is not maintained in the case of the proposed expansion procedure (40). However this is not a basic matter of concern. The expanded system is only an artificial mathematical construction and the crucial issue is that, with the proposed complementary matrices, the inclusion principle is guaranteed. Moreover, the expanded system (40) has a structure with subsystems of equal dimensions as the original overlapping subsystems. This is a structural constraint that is preserved in the proposed approach. The benefit of this approach may be recognized in a significant reduction of the performance index satisfying simultaneously conditions on the original information structure constraints of controllers.

5 Conclusion

The Inclusion Principle has been specialized for a quadratic optimal control design for general discrete-time LTV systems. The generalized structure of complementary matrices has been presented for this class of systems. Explicit conditions on their structure have been proved, including conditions on contractibility of controllers. In the case of continuous-time LTV systems, these structures have been recently available only for the specific class of commutative systems. In contrast to this case, it has been shown that the generalized structure of complementary matrices can be obtained without any restrictions for general discrete-time LTV systems. A guideline for selection of complementary matrices is included when considering a particular set of expansion/contraction transformation matrices. The effectiveness of the use of these generalized structures has been demonstrated on an illustrative example.

6 Acknowledgements

This work was supported in part by the DURSI under Grant PIV, by the ASCR under Grant A2075304 and by the CICYT under Grant DPI2002-04018-C02-01.

7 References

[1] L. Bakule, and J. Rodellar, Decentralized control and overlapping decomposition of mechanical systems. Part 1: System decomposition. Part 2: Decentralized stabilization, *Internat. J. of Control*, **61**, (1995) 559-587.

[2] L. Bakule, J. Rodellar, and J.M. Rossell, Structure of expansion-contraction matrices in the inclusion principle for dynamic systems, *SIAM J. Matrix Anal. Appl.*, **21(4)**, (2000) 1136-1155.

[3] L. Bakule, J. Rodellar, and J.M. Rossell, Generalized selection of complementary matrices in the inclusion principle, *IEEE Trans. Automat. Control*, **45(6)**, (2000) 1237-1243.

[4] L. Bakule, J. Rodellar, and J.M. Rossell, Overlapping quadratic optimal control of linear time-varying commutative systems, *SIAM J. Control Optim.*, **40(5)**, (2002) 1611-1627.

[5] L. Bakule, J. Rodellar, and J.M. Rossell, Overlapping LQ control of discrete-time time-varying systems, Large Scale Systems: Theory and Applications 2001, Proc. IFAC, Pergamon, Oxford, (2002) 425-430.

[6] M. Ikeda, and D.D. Šiljak, Overlapping decompositions, expansions and contractions of dynamic systems, *Large Scale Systems*, **1**, (1980) 29-38.

[7] M. Ikeda, D.D. Šiljak, and D.E. White, An inclusion principle for dynamic systems, *IEEE Trans. Automat. Control*, **29**, (1984) 244-249.

[8] M. Ikeda, D.D. Šiljak, and D.E. White, Overlapping decentralized control of linear time-varying systems, *Advances in Large Scale Systems, JAI Press*, **1** (1984) 93-116.

[9] H. Kwakernaak, and R.Sivan, Linear Optimal Control Systems, Wiley, New York, 1972.

[10] W.J. Rugh, Linear System Theory (2nd ed.), Prentice Hall, Upper Saddle River, New Jersey, 1996.

[11] M.E. Sezer, and D.D. Šiljak , Nested epsilon decomposition of linear systems: weakly coupled and overlapping blocks, *SIAM J. Matrix Anal. Appl.*, **12(3)**, (1991) 521-533.

[12] D.D. Šiljak, Decentralized Control of Complex Systems, Academic Press, New York, 1991.

[13] D.D. Šiljak, S.M. Mladenović, and S. Stanković, Overlapping decentralized observation and control of a platoon of vehicles, *Proc. of the American Control Conference, San Diego, California*, (1999) 4522-4526.

[14] S.S. Stanković, and D.D. Šiljak, Inclusion principle for time-varying systems, *SIAM J. Control Optim.*, **42(1)**, (2003) 321-341.

email:journal@monotone.uwaterloo.ca
http://monotone.uwaterloo.ca/∼journal/