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CONVEXITY OF THE REACHABLE SET OF NONLINEAR SYSTEMS UNDER L_2 BOUNDED CONTROLS

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Abstract. Recently [7, 8] the new convexity principle has been validated. It states that a nonlinear image of a small ball in a Hilbert space is convex, provided that the map is $C^{1,1}$ and the center of the ball is a regular point of the map. We demonstrate how the result can be exploited to guarantee the convexity of the reachable set for some nonlinear control systems with L_2-bounded control. This provides existence and uniqueness of the solution in some optimal control problems as well as necessary and sufficient optimality conditions and effective iterative methods for finding the solution.
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1 Introduction

Convexity plays a key role in functional analysis, optimization and control theory. For instance, if a mathematical programming problem is convex, then necessary optimality conditions coincide with sufficient ones, duality theorems hold and effective numerical methods can be constructed [12]. However, convex problems are just a small island in the ocean of non convex ones.

In the present paper we describe the technique, which is useful for establishing convexity in nonlinear control problems. It is based on the recent result [7, 8], asserting convexity of a nonlinear image of a small ball in a Hilbert space. This result is addressed in Section 2, while all other Sections deal with its applications to nonlinear control systems. Namely, in Section 3 we prove the convexity of the reachable set for nonlinear systems provided that the linearized system is controllable while the control is bounded and small enough in L_2 norm. For linear systems the explicit description of the reachable set with L_2-bounded control goes back to Kalman [4]. Lee and Markus [6] (see also [5]) proved that controllability of the linearized system implies local controllability of the nonlinear system. However we don't know any result on the convexity of the reachable set for nonlinear case; some related ideas can be found in the earlier paper of the author [9]. In Section 4 we apply the statements of the previous Section to get necessary and sufficient optimality conditions in related optimal control problems and provide a simple iterative algorithm to solve the problems. An example (a mathematical pendulum) is considered in Section 5.

2 The convexity principle

Let X, Y be two Hilbert spaces, let $f : X \to Y$ be a nonlinear map with Lipschitz derivative on a ball $B(a, r) = \{x \in X : ||x - a|| \le r\}$, thus

$$||f'(x) - f'(z)|| \le L||x - z|| \quad \forall x, z \in B(a, r). \tag{1}$$

Suppose that a is a regular point of f, i.e. the linear operator $f'(a)$ maps X onto Y; then there exists $\nu > 0$ such that

$$||f'(a)^* y|| \ge \nu ||y|| \quad \forall y \in Y. \tag{2}$$

For instance, for X, Y finite dimensional: $X = \mathbf{R}^n, Y = \mathbf{R}^m$, this condition holds if rank$f'(a) = m$; for this case $\nu = \sigma_1(f'(a))$ — the least singular value of $f'(a)$.

Theorem 1 *If (1), (2) hold and* $\varepsilon < \min\{r, \nu/(2L)\}$, *then the image of a ball* $B(a, \varepsilon) = \{x \in X : ||x - a|| \le \varepsilon\}$ *under the map* f *is convex, i.e.* $F = \{f(x) : x \in B(a, \varepsilon)\}$ *is a convex set in* Y. *Moreover, the set is strictly convex and its boundary is generated by boundary points of the ball:* $\partial F \subset f(\partial B(a, \varepsilon))$.

The rigorous proof of Theorem 1 is involved and can be found in [7, 8] while its idea is very simple. The ball $B(a, \varepsilon)$ is strongly convex, thus its image under the linear map $f'(a)$ is strongly convex as well. But it can not loose convexity for a nonlinear map f, which is close enough to its linearization. The same reasoning explains that the result can not be extended to an arbitrary Banach space, where a ball is not strongly convex. However the extension of the principle to uniformly convex Banach spaces (such as $L_p, 1 < p < \infty$) is an open problem. On the other hand, the result remains true, if we replace the ball by any other strongly convex set (e.g. by a nondegenerate ellipsoid). For a particular case $f : \mathbf{R}^n \to \mathbf{R}^n$ Theorem 1 has been extended in [1] for strictly convex (not necessarily strongly convex) sets. Note that smoothness assumptions of Theorem 1 can not be seriously relaxed. For instance, A.Ioffe constructed a counterexample with f continuously differentiable but not in $C^{1,1}$. Then the result is false. Theorem 1 relates to so called mapping theorems, the recent survey of investigations in this field can be found in [3]. The simplest mapping teorem by Graves [2] claims that a ball can be inscribed in F, and its radius linearly depends on ε; it is true in Banach (not necesserily Hilbert) spaces and under weeker assumptions on smoothness of f.

Now we describe how the boundary of the image set ∂F can be effectively generated. Consider the optimization problem for $c \in Y$

$$\min_{||x-a|| \le \varepsilon} (c, f(x)) = \min_{f \in F} (c, f). \tag{3}$$

If F is strictly convex, then the solution of this problem for arbitrary $c \neq 0$ is a boundary point of F and vice verse, any boundary point of F is a solution of (3) for some $c \neq 0$. The necessary and sufficient condition for a point x^* to be a solution of (3) is

$$x^* = a - \varepsilon \frac{f'(x^*)^* c}{||f'(x^*)^* c||}. \tag{4}$$

A simple iterative method to solve equation (4) reads:

$$x^{k+1} = a - \varepsilon \frac{f'(x^k)^* c}{||f'(x^k)^* c||}. \tag{5}$$

We summarize the technique for constructing boundary points of F as follows.

Theorem 2 *Under assumptions of Theorem 1 for any $c \neq 0$*

a)The solution x^* of (3) exists and is unique, it belongs to ∂F, $||x^* - a|| = \varepsilon$ and the necessary and sufficient optimality condition (4) holds. Moreover the set of solutions of (3) for all $c \in Y, ||c|| = 1$ coincides with ∂F.

b)Method (5) converges with linear rate of convergence for any $x^0 \in B(a, \varepsilon)$:

$$||x^k - x^*|| \leq q^k ||x^0 - x^*||, \quad q = O(\varepsilon) = \frac{\varepsilon L}{\nu - \varepsilon L} < 1. \tag{6}$$

The proof follows immediately from the strict convexity of F provided by Theorem 1 and necessary and sufficient optimality condition for minimization of a differentiable function with nonvanishing gradient on a ball; see details in [7, 8].

Another characterization of the boundary points of the set F (with ε, not prescribed in advance), can be obtained by the following result.

Theorem 3 *If (1), (2) hold and $d \in Y, ||d - f(a)||$ is small enough then*

a) d belongs to the boundary of F with $\varepsilon = \min_{f(x)=d} ||x - a||$.

b) The solution x^* of the optimization problem

$$\min_{f(x)=d} ||x - a|| \tag{7}$$

exists, is unique and satisfies necessary and sufficient optimality condition: there exists $y \in Y$ such that

$$x^* = a + f'(x^*)^* y \tag{8}$$

Say it another way, find

$$g(y) = \min_{f(x)=y} ||x - a|| \tag{9}$$

then F is the level set of this function $F = \{y : g(y) \le \varepsilon\}$ while $\partial F = \{y : g(y) = \varepsilon\}$.

Note that in contrast with Theorem 1 there is no explicit bound for "close enough" distance in Theorem 3.

In many cases the conditions of Theorem 1 can be effectively checked, and the radius ε of the ball can be estimated. One of such examples is a quadratic transformation.

Example. Let $x \in \mathbf{R}^n$ and $f(x) = (f_1(x), ...f_m(x))^T$ where $f_i(x)$ are quadratic functions:

$$f_i(x) = (1/2)(A_i x, x) + (a_i, x), \quad A_i = A_i^T \in \mathbf{R}^{n\times n}, a_i \in \mathbf{R}^n, \quad i = 1, ...m. \tag{10}$$

Take $a = 0$, that is $B = \{x : ||x|| \le \varepsilon\}$. Then $f_i'(x) = A_i x + a_i$ and (1) is satisfied on $\mathbf{R}^n$ with $L = (\sum_{i=1}^m ||A_i||^2)^{1/2}$ where $||A_i||$ stands for the spectral norm of matrices A_i. Consider the matrix A with columns a_i: $A = (a_1|a_2|...|a_m)$. Then $f'(0)^T y = Ay$, and if rank$A = m$, then (2) holds with $\nu = \sigma_1(A)$ — the minimal singular value of A, that is $\nu = (\min \lambda_1(A^T A))^{1/2}$, where λ_1 is the minimal eigenvalue of the corresponding matrix. Hence, Theorem 1 implies:

Proposition 1 *If $\varepsilon < \nu/(2L)$, then the image of the ball B under the map (10) is convex:*

$$F = \{f(x) : ||x|| \le \varepsilon\}$$

is a convex set in $\mathbf{R}^m$.

This is in a sharp contrast with the results on images of arbitrary balls under quadratic transformations, where the convexity can be validated [10] just under very restrictive assumptions.

For instance, let $n = m = 2$ and

$$f_1(x) = x_1 x_2 - x_1, f_2(x) = x_1 x_2 + x_2. \tag{11}$$

Then the estimates above guarantee that F is convex for $\varepsilon < \varepsilon^* = 1/(2\sqrt{2}) \approx 0.3536$. It can be proved directly for this case that F is convex for $\varepsilon \le \varepsilon^*$ and looses convexity for $\varepsilon > \varepsilon^*$. Thus the estimate provided by Proposition 1 is tight for this example.

Figure 1 shows the images of the ε-discs $\{x \in \mathbf{R}^2 : ||x|| \le \varepsilon\}$ under the mapping (11) for various values of ε.

Other examples, related to optimization problems and to linear algebra can be found in [7, 8, 11]. In what follows, we focus on control applications of the convexity principle.

3 Convexity of the reachable set

A general nonlinear control system

$$\dot{x} = \phi(x, u, t), x \in \mathbf{R}^n, u \in \mathbf{R}^m, 0 \le t \le T, x(0) = b \tag{12}$$

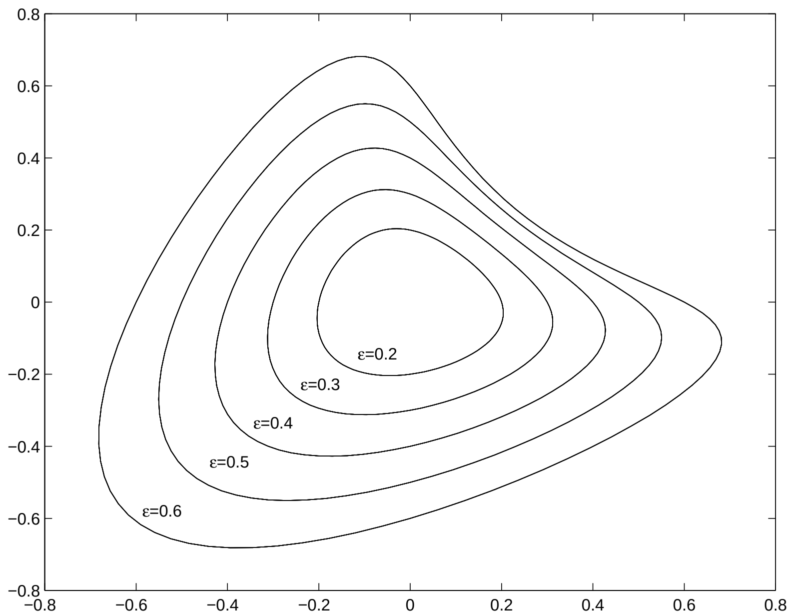

Figure 1: Image of a quadratic map

with L_2-bounded control

$$u \in U = \{u : ||u||^2 = \int_0^T |u(t)|^2 dt \le \varepsilon^2\} \tag{13}$$

defines a reachable set

$$F = \{x(T) : x(t) \text{ is a solution of } (12), u \in U\}. \tag{14}$$

¿From here on we denote finite dimensional Euclidean norms of vectors and matrices as $|.|$ (e.g.$|u(t)|$) while L_2 norm is denoted $||u||$. Our aim is to prove that under some assumptions the reachable set is convex for ε small enough. We do this relying on Theorem 1 with $X = L_2, Y = \mathbf{R}^n, a = 0, f : u \to x(T)$. Thus we should check conditions of the Theorem for this situation. Consider the linearized system

$$\dot{z} = \phi_x(x_0, 0, t)z + \phi_u(x_0, 0, t)u, \quad y(0) = 0 \tag{15}$$

here x_0 is the nominal trajectory, corresponding to nominal control $u_0 = 0$:

$$\dot{x_0} = \phi(x_0, 0, t), \quad x_0(0) = b.$$

Now we formulate some technical assumptions to verify conditions of Theorem 1.

A 1 *The function $\phi(x,u,t)$ is continuous in x,u, measurable in t and satisfies the growth condition for all $x,u,0 \le t \le T$:*

$$|\phi(x,u,t)| \le k_1 + k_2|x| + k_3|u|^2.$$

In what follows all constants will be denoted k_i. The above growth condition is restrictive (for instance, a term like xu is not allowed in ϕ). Under above assumption we can guarantee the boundedness of solutions of (12).

Lemma 1 *If A1 holds, then solutions of (12) are bounded:*

$$|x(t)| \le k_0, \quad 0 \le t \le T$$

for all $u \in U$.

Proof We have from (12)

$$|x(t)| = |b + \int_0^t \phi(x(\tau),u(\tau),\tau)d\tau| \le |b| + k_1 T + k_3\varepsilon^2 + k_2 \int_0^t |x(\tau)|d\tau|.$$

Applying Gronwall lemma, we get the desired inequality.

Introduce the new growth condition:

A 2 *For all $|x| \le k_0, |x+\bar{x}| \le k_0, u, u+\bar{u}, 0 \le t \le T$*

$$|\phi(x,u,t) - \phi(x+\bar{x},u+\bar{u},t)| \le k_4|\bar{x}| + k_5|\bar{u}| + k_6|\bar{u}^2|.$$

Then we can estimate the distance between two solutions of the equation (12).

Lemma 2 *If A1, A2 hold, then for two solutions $x(t), x(t)+\bar{x}(t)$ of (12) corresponding to two controls $u(t), u(t)+\bar{u}(t), u \in U, u+\bar{u} \in U, \bar{u} \in U$ we have*

$$|\bar{x}(t)| \le k_7||\bar{u}||, \quad 0 \le t \le T.$$

Proof Indeed

$$\dot{\bar{x}} = \phi(x+\bar{x},u+\bar{u},t) - \phi(x,u,t), \quad \bar{x}(0) = 0.$$

Integrating and applying A2 we obtain

$$|x(t)| \le k_4 \int_0^t |x(\tau)|d\tau + k_5||\bar{u}||t^{1/2} + k_6||\bar{u}||^2 \le k_4 \int_0^t |x(\tau)|d\tau + (k_5 T^{1/2} + k_6\varepsilon)||\bar{u}||.$$

Now Gronwall lemma implies the assertion of the Lemma.

Under more tight conditions we can estimate how close are the solutions of the perturbed system and the linearized system.

A 3 *Function $\phi(x,u,t)$ is differentiable in x,u and for all $|x| \le k_0, |x+\bar{x}| \le k_0, u, u+\bar{u}, 0 \le t \le T$*

$$|\phi_x(x,u,t)| \le k_8 \tag{16}$$
$$|\phi(x,u,t) - \phi(x+\bar{x}, u+\bar{u}, t) - \phi_x(x,u,t)\bar{x} - \phi_u(x,u,t)\bar{u}| \le k_9|\bar{x}|^2 + k_{10}|\bar{u}^2|.$$

Consider the linearized system

$$\dot{z} = \phi_x(x,u,t)z + \phi_u(x,u,t)\bar{u}, \quad z(0) = 0 \tag{17}$$

where $x(t)$ is a solution of (12) for a control $u \in U$ while $\bar{u}(t)$ is a perturbation of control. Linearized system (15) is a particular case of (17) for nominal control $u = 0$.

Lemma 3 *If $x(t), x(t)+\bar{x}(t)$ are two solutions of (12) corresponding to two controls $u(t), u(t)+\bar{u}(t), u \in U, u+\bar{u} \in U, \bar{u} \in U$ while $z(t)$ is the solution of the linearized system (17), then under A1- A3 the following estimate holds*

$$|\bar{x}(t) - z(t)| \le k_{11}||\bar{u}||^2, \quad 0 \le t \le T.$$

This result ensures the differentiabilty of the map $f : u \to x(T)$. Indeed, consider $z(T)$ - the solution of (17), then $z(T)$ is the linear map of $\bar{u} : z(T) = \mathcal{L}\bar{u}$. On the other hand the result of Lemma 3 can be rewritten (for $t = T$) as

$$|f(u+\bar{u}) - f(u) - \mathcal{L}\bar{u}| \le ||\bar{u}||^2.$$

This means that $f(u)$ is differentiable and $f'(u)\bar{u} = z(T)$.

The next asumption guarantees that $f \in C^{1,1}$.

A 4 *For all $|x| \le k_0, u, 0 \le t \le T$ partial derivatives $\phi_x(x,u,t), \phi_u(x,u,t)|$ are Lipschitz in x,u.*

Lemma 4 *Under A1 - A4 the map $f : u \to x(T)$ is differentiable with Lipshitz derivative on U.*

We skip the proof, it follows the same lines as before.

The last assumption implies the regularity of of the map f; it states that $f'(0)\bar{u} = z(T)$ coincides with the whole space $\mathbf{R^n}$.

A 5 *The linearized system (15) is controllable.*

Combining all the above lemmas, we see that the assumptions of Theorem 1 hold true for our map, thus we validated the main result of the paper:

Theorem 4 *If A1 -A5 are satisfied and $\varepsilon > 0$ is small enough, then the reachable set (14) is convex.*

Let us discuss the result. First, we do not provide the explicit bound for $\varepsilon > 0$ which guarantees convexity. In principle it can be done: we can estimate all constants introduced above (for instance, in Lemma 1 we get $k_0 \le (|b| + k_1 T + k_3 \varepsilon^2)\exp(k_2 T)$, thus we obtain L in Theorem 1, and we can estimate ν via controllability Grammian for equation (15) (see analysis of the linear case below). However such bound would be conservative and of little value. Second, most of the assumptions A1 − A4 are just natural smoothness and growth conditions and they are not very restrictive. The main exception is the boundedness condition $|\phi_x(x,u,t)| \le k$. It does not permit to have terms like $h(x)u$ in $\phi(x,u,t)$. However, terms like $h(x)g(u)$ with $g(u)$ bounded or $a(t)u + b(t)u^2$ are suitable.

The general description of the reachable set for linear systems has been provided by Kalman and coauthors [4] - it is an ellipsoid even without controllability assumption. Of course this result is obvious - the reachable set is a linear finite-dimensional image of L_2 ball and hence is an ellipsoid. The explicit description of this ellipsoid for controllable case, given in [4], writes as follows. Let

$$\dot{x} = A(t)x + B(t)u, \quad x(0) = 0 \tag{18}$$

and

$$F = \{x(T) : x(t) \text{ is a solution of (18)}, \int_0^T |u(t)|^2 dt \le \varepsilon^2\}. \tag{19}$$

Then

$$F = \{x : x^T W(T)^{-1} x \le \varepsilon^2\} \tag{20}$$

where $W(T)$ is the controllability Grammian:

$$W(T) = \int_0^T \Phi(t)B(t)B^T(t)\Phi^T(t)dt \tag{21}$$

which is assumed to be nondegenerate, while $\Phi(t)$ is the fundamental matrix of (18).

4 Optimal control

Relying on the convexity result, we are able to analyse some optimal control problems. Consider the terminal optimization problem for system (12) subject to constraints (13):

$$\min(c, x(T)) \tag{22}$$

$$\dot{x} = \phi(x,u,t), x \in \mathbf{R}^n, u \in \mathbf{R}^m, 0 \le t \le T, x(0) = b$$

$$\int_0^T |u(t)|^2 dt \le \varepsilon^2,$$

where $c \in \mathbf{R}^n \neq 0$. Then this optimal control problem is equivalent to finite-dimensional one:

$$\min_{x \in F}(c, x)$$

which is convex under assumptions of Theorem 4. This leads to the following result.

Theorem 5 *If A1- A5 are satisfied and $\varepsilon > 0$ is small enough, then:*
a) The solution x^*, u^* of the problem (22) exists and is unique, $||u^*|| = \varepsilon$;
b) It satisfies the necessary and sufficient optimality condition

$$u^*(t) = \frac{-\varepsilon \phi_u^T(x^*, u^*, t)\psi(t)}{\int_0^T |\phi_u^T(x^*, u^*, \tau)\psi(\tau)|^2 d\tau},$$

where $\psi(t)$ is the solution of the adjoint system:

$$\dot{\psi} = -\phi_x^T(x^*, u^*, t)\psi(t), \quad \psi(T) = -c;$$

c) The iterative method

$$\begin{aligned} \dot{x}^k &= \phi(x^k, u^k, t), \quad x^k(0) = b \\ \dot{\psi}^k &= -\phi_x^T(x^k, u^k, t)\psi^k(t), \quad \psi^k(T) = -c \\ u^{k+1}(t) &= \frac{-\varepsilon \phi_u^T(x^k, u^k, t)\psi^k(t)}{\int_0^T |\phi_u^T(x^k, u^k, \tau)\psi^k(\tau)|^2 d\tau}, \end{aligned} \tag{23}$$

converges to the solution with linear rate: $||u^k - u^*|| \leq Cq^k, q < 1$.

Proof It is a direct corollary of Theorem 2 and Theorem 4.

This result can be easily extended to more general optimization problem where the linear function $(c, x(T))$ is replaced with nonlinear Lipschitz differentiable function $h(x(T))$ with the only assumption $h'(x_0(T)) \neq 0$. Theorem 4 remains true with c replaced by $h'(x^*(T))$ in optimality conditions and by $h'(x^k(T))$ in the iterative method.

On the other hand, instead of (22) we can consider another optimal control problem with fixed terminal point and quadratic performance:

$$\begin{gathered} \min \int_0^T |u(t)|^2 dt \\ \dot{x} = \phi(x, u, t), \quad 0 \leq t \leq T, x(0) = b, x(T) = d. \end{gathered} \tag{24}$$

This can be covered by Theorem 3 which immediately implies the following result.

Theorem 6 *If A1- A5 are satisfied and $||d - x_0(T)||$ is small enough, then:*
a) The solution x^*, u^* of the problem (24) exists and is unique,

b) It satisfies the necessary and sufficient optimality condition: there exist $y \in \mathbf{R^n}$ *such that*

$$u^*(t) = \phi_u^T(x^*, u^*, t)\psi(t)$$

where $\psi(t)$ *is the solution of the adjoint system:*

$$\dot{\psi} = -\phi_x^T(x^*, u^*, t)\psi(t), \quad \psi(T) = y.$$

¿From the above optimality conditions we can conclude that if we assume additionally that

$$|\phi_u(x, u, t)| \le k$$

for all all bounded x, all u and all $0 \le t \le T$ then optimal controls in both problems (22), (24) are continuous and bounded. This means that the reachable set in (22) is the same as in similar problem with an extra constraint $|u(t)| \le M, 0 \le t \le T$ for M large enough. This allows to relax the most restrictive condition (16). Indeed it suffices to assume that

$$|\phi_x(x, u, t)| \le k_8 \tag{25}$$

for all *bounded* x, u. Note that A1 must be satisfied for *all* x, u. Then, for instance, affine in u equation

$$\phi(x, u, t) = h_0(x) + \sum_{i=0}^{m} u_i h_i(x), \quad |h_0(x)| \le k_{12} + k_{13}|x|, |h_i(x)| \le k_{14} + k_{15}\sqrt{|x|} \tag{26}$$

is acceptable.

5 Example - a pendulum

Consider a pendulum forced by small energy control:

$$\begin{gathered} \dot{x}_1 = x_2, \quad x_1(0) = 0 \\ \dot{x}_2 = -\omega^2 \sin x_1 + u, \quad x_2(0) = 0 \\ \int_0^T |u(t)|^2 dt \le \varepsilon^2. \end{gathered} \tag{27}$$

For simplicity of calculations we accept zero initial conditions, thus nominal solution is $u_0(t) = 0, x_0(t) = 0$. Our aim is to describe the reachable set for this problem. First of all we can easily check that growth and smoothness assumptions A1-A4 are satisfied. The linearized system along the nominal trajectory

$$\begin{gathered} \dot{x}_1 = x_2, \quad x_1(0) = 0 \\ \dot{x}_2 = -\omega^2 x_1 + u, \quad x_2(0) = 0 \end{gathered} \tag{28}$$

is obviously controllable. Thus we conclude that for ε small enough the reachable set of the system (27) is convex.

Before constructing F itself, we describe the reachable ellipsoid for the linearized system (28) in accordance with (20). In our case

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, e^{At} = \begin{pmatrix} \cos\omega t & (1/\omega)\sin\omega t \\ -\omega\sin\omega t & \cos\omega t \end{pmatrix},$$

and calculation of the Grammian provides

$$W(T) = \begin{pmatrix} \frac{T}{2\omega^2} - \frac{\sin 2\omega T}{4\omega^3} & \frac{1-\cos 2\omega T}{4\omega^2} \\ \frac{1-\cos 2\omega T}{4\omega^2} & \frac{T}{2} + \frac{\sin 2\omega T}{4\omega} \end{pmatrix}.$$

If we choose $T = 2\pi/\omega$, then $W(T)$ becomes diagonal and the reachable set is the ellipsoid

$$F = \{x : 4\omega^3 x_1^2 + 4\omega x_2^2 \le \pi\varepsilon^2\}.$$

For $\omega = 1$ this is a disc.

Now, to draw F for the nonlinear pendulum (27) we can apply algorithm (23) for various $c \in \mathbf{R^2}, |c| = 1$. Thus c is one parametric family, for instance we can take $c = (\sin\theta, \cos\theta)^T, 0 \le \theta \le 2\pi$. However, on this way we can construct F only if it is convex. To handle the case of large ε, when F looses the convexity, another technique was applied. Concider optimization problem of the type (24):

$$\begin{aligned} &\min \int_0^T |u(t)|^2 dt, \qquad (29)\\ \dot{x}_1 &= x_2, \quad x_1(0) = 0,\ x_1(T) = d_1, \\ \dot{x}_2 &= -\omega^2 \sin x_1 + u, \quad x_2(0) = 0,\ x_2(T) = d_2. \end{aligned}$$

Its solution satisfies optimality condition

$$u(t) = \psi_2(t)$$

where ψ is the solution of the adjoint system:

$$\begin{aligned} \dot{\psi}_1 &= \omega^2 \cos(x_1)\psi_2, \qquad (30)\\ \dot{\psi}_2 &= -\psi_1. \end{aligned}$$

Thus to construct the reachable set for (27) we solve initial point problem for ODE

$$\begin{aligned} \dot{x}_1 &= x_2, \quad x_1(0) = 0, \qquad (31)\\ \dot{x}_2 &= -\omega^2 \sin x_1 + \psi_2, \quad x_2(0) = 0, \\ \dot{\psi}_1 &= \omega^2 \cos(x_1)\psi_2, \quad \psi_1(0) = a_1, \\ \dot{\psi}_2 &= -\psi_1, \quad \psi_2(0) = a_2 \end{aligned}$$

with some initial conditions a and calculate $x(T)$ and $v = \int_0^T |\psi_2(t)|^2$. By gridding a we calculate the function of two variables $v(x(T))$ on some grid. The level sets of this function are exactly the boundary of F, corresponding to $v = \varepsilon^2$ (see Theorems 3, 6). The results of calculations are presented on Figure 2 for $\varepsilon^2 = 0.6 : 0.1 : 1.0$. It can be seen as F looses the convexity when ε increases.

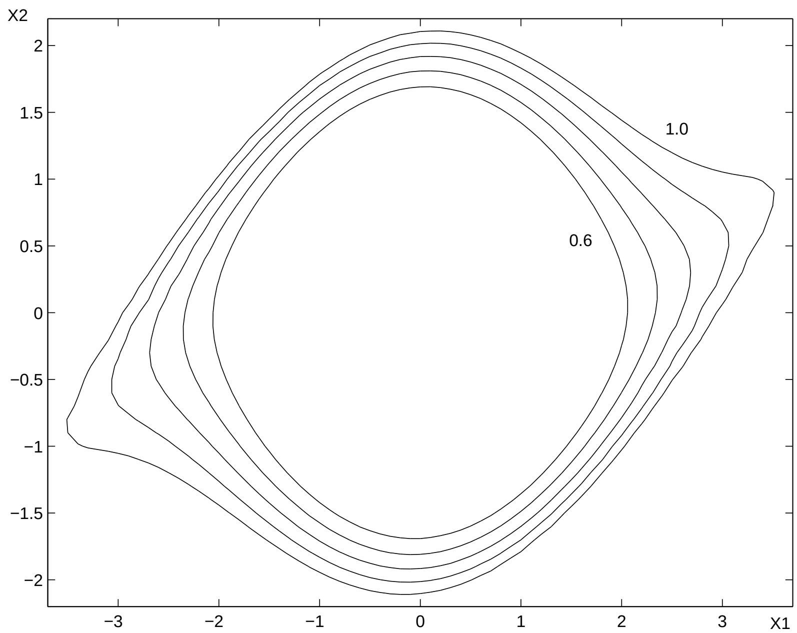

Figure 2: Reachable set for the pendulum

6 Conclusions

The new "image convexity" principle is a promising tool for analysis of various control problems. It validates the convexity of the reachable set for systems with L_2-bounded controls, guarantees uniqueness of the solution of energy-optimal controls, provide necessary and sufficient conditions and efficient numerical methods for these controls. Similar technique can be applied to some other optimal control problems with more general performance index or with discrete time.

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References

[1] S.V.Emelyanov, S.K.Korovin, N.A.Bobylev, On convexity of images of convex sets under smooth transformations, *Doklady RAN,* **385**, No 3, (2002), 302–304.

[2] L.M.Graves, Some mapping theorems, *Duke Math, J.*, **17**, (1950), 111–114.

[3] A.Ioffe, Metric regularity and subdifferential calculus, *Russ. Math. Surveys*, **55**, No 3, (2000), 501–556.

[4] R.E.Kalman, Y.C.Ho, K.S.Narendra, Controllability of linear dynamical systems, *Contributions to Differential Equations*, **1**, No 2, (1963), 189–213.

[5] E.B.Lee, L.Markus, Foundations of Optimal Control Theory, John Wiley, New York, 1970.

[6] L.Markus, E.B.Lee, On the existence of optimal controls, *J. Basic Engr. (Trans. ASME)*, **84D**, (1962), 1–20.

[7] B.Polyak, Convexity of nonlinear image of a small ball with applications to optimization, *Set-Valued Analysis*, **9**, No 1/2, (2001), 159–168.

[8] B.Polyak, Local programming, *Comp. Math. and Math. Phys.,* **41**, No. 9, (2001), 1259–1266.

[9] B.T.Polyak, On the theory of nonlinear control problems, *Vestnik Mosc. Univ., Ser. Math., Mech.*, No 2, (1968), 30–40 (in Russian).

[10] B.T.Polyak, Convexity of quadratic transformations and its use in control and optimization, *Journ. Optim. Th. and Appl.,* **99**, No 3, (1998), 553–583.

[11] B.T.Polyak, The convexity principle and its applications, *Bull. Brazilian Math. Soc.,* **34**, No 1, (2003), 1–17.

[12] R.T.Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.