Lesson 10 Game Theory

1: In a table, there are k heaps of coins, where the heaps contain $a_1, a_2, \ldots, a_k > 0$ coins, respectively. Two players A and B take turns, where in each turn the current player removes several (at least one) coins from a heap. The game ends when all coins have been removed, and the last player to make a move wins. Assuming A moves first and both players use optimal strategies, determine the outcome of the game.

2: (IMO 2009 C1) Let n and k be positive integers. Given n cards placed on the table from left to right, each having one gold side and one black side. Two players A and B take turns, where in each turn the current player chooses a block of k consecutive cards, the leftmost of which is showing gold, and flips all k cards over. The game ends when there are no more legal moves, and the last player to move wins.

Prove that the game must eventually end. Assuming A moves first and both players use optimal strategies, determine the outcome of the game.

3: (Putnam 2020 B2) Let $k < n$ be positive integers. Initially, there are k pegs in a line of n holes, with the pegs occupying the k leftmost holes. Two players A and B take turns, where in each turn the current player moves one peg to a *vacant* hole on the right hand side with respect to the current position of the peg to be moved. The game ends when the k pegs occupy the k rightmost holes, and the last player to make a move wins. Assuming A moves first, for which k and n does A have a winning strategy?

4: (IMO 2014 C2) Initially, we are given 2^m sheets of paper, with the number 1 written on each sheet. Consider the following operation: choose two distinct sheets, say with numbers a and b written on them, and replace the two numbers with $a + b$ on both sheets. Prove that, after $m2^{m-1}$ such operations, the sum of the numbers on all the sheets is at least 4^m.

5: (IMO 2017 N2) Let p be a prime number. Initially, there are p variables $a_0, a_1, \ldots, a_{p-1}$ with unknown values. Two players A and B play the following game, making moves alternately: in each move, the current player chooses a pair (i, a_i), where $i \in \{0, 1, \ldots, p-1\}$ and $a_i \in \{0, 1, 2, \ldots, 9\}$, such that the index i has not been chosen in any previous moves. (For example, if $(1, 8)$ has been chosen in the first move, then $(1, 9)$ cannot be chosen in any subsequent moves.) The game ends after each indices $i \in \{0, 1, \ldots, p-1\}$ have been chosen. A wins if at the end of the game, the following integer is divisible by p:

$$M = a_0 + 10a_1 + 10^2a_2 + \ldots + 10^{p-1}a_{p-1}.$$

Otherwise, B wins.

Suppose that A has the first move. Prove that A has the winning strategy.

6: (IMO 2015 C4) Let n be a positive integer. Two players A and B play the following game, where in each turn, the current player chooses a positive integer less than or equal to n that has not been chosen in any previous turn. In addition, both players cannot choose a number consecutive to any positive integer that the same player has chosen before. (Two integers a and b are consecutive if $|a - b| = 1$.)

The game ends in a draw if all positive integers at most n has been chosen; otherwise the player who cannot move loses. Assuming that A takes the first turn and that both players use optimal strategies, determine the outcome of the game.

7: (IMO 2020 C8) In a blackboard, 2020 copies of the integer 1 is written. Two players A and B play the following game, where in each round, A removes two integers from the blackboard, say x and y, and then B writes exactly one copy of either $x + y$ or $|x - y|$ on the blackboard. The game terminates when either:

- there exists an integer on the blackboard that is greater than the sum of all the other integers on the blackboard,
- all integers on the blackboard are zero.

Let N be the number of integers written on the blackboard at the end of the game. A's goal is to maximize N, while B's goal is to minimize N. Determine the value of N, assuming both players use optimal strategies.

8: (Putnam 2017 A5) Let n be a positive integer. Given a deck of n cards labelled from 1 to n, in a shuffled state. Three players A, B, and C play a game, where in each turn the current player chooses one card at random from the deck, say with label k, then removes every card with label greater than or equal to k and reshuffles the deck. The game ends when a player draws 1, and this player wins. The three players take turns in the following ordering: A, B, C, A, B, C, A, $\ldots$.

Show that, for each of the three players, there exists infinitely many positive integers n for which the player has the highest probability of winning the game.

9: (Putnam 2022 A5) Alice and Bob play a game on a board consisting of one row of 2022 consecutive squares. They take turns placing tiles that cover two adjacent squares, with Alice going first. By rule, a tile must not cover a square that is already covered by another tile. The game ends when no tile can be placed according to this rule. Alice's goal is to maximize the number of uncovered squares when the game ends; Bob's goal is to minimize it. What is the greatest number of uncovered squares that Alice can ensure at the end of the game, no matter how Bob plays?

Putnam Problems Involving Games

1: (1997 B2) Players $1, 2, 3, \ldots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

2: (1995 B5) A game starts with four heaps of beans, containing 3,4,5 and 6 beans. The two players move alternately. A move consists of taking either

(a) one bean from a heap, provided at least two beans are left behind in that heap, or

(b) a complete heap of two or three beans.

The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

3: (1993 B2) Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

4: (1989 A4) If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both 1/2. A game is finite if with probability 1 it must end in a finite number of moves.)

5: (2002 A4) In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

6: (2006 A2) Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if n = 17, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

7: (2008 A2) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

8: (2009 B2) A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

9: (2010 B3) There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

10: (2013 B6) Let $n \geq 1$ be an odd integer. Alice and Bob play the following game, taking alternating turns, with Alice playing first. The playing area consists of n spaces, arranged in a line. Initially all spaces are empty. At each turn, a player either

(a) places a stone in an empty space, or

(b) removes a stone from a nonempty space s, places a stone in the nearest empty space to the left of s (if such a space exists), and places a stone in the nearest empty space to the right of s (if such a space exists).

Furthermore, a move is permitted only if the resulting position has not occurred previously in the game. A player loses if he or she is unable to move. Assuming that both players play optimally throughout the game, what moves may Alice make on her first turn?

11: (2014 B5) In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n \times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(a) A player cannot choose an element that has been chosen by either player on any previous turn.

(b) A player can only choose an element that commutes with all previously chosen elements.

(c) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

Hints

1: Consider breaking up each heap into subheaps of sizes $1, 2, 4, 8, \ldots$. A will aim to achieve the state where there are even number of subheaps of every size.

2: To prove that the game eventually ends, consider representing the game state as a binary string. To find the outcome of the game, consider labelling the m^{th} card from the right for every m divisible by k.

3: A can make a move to change (k, n) to $(k-1, n-2)$ or $(k-1, n-1)$ or $(k, n-1)$. In order for A to lose, all three possibilities should result in a first player win. This suggests that A loses when k has a certain parity and n has a certain parity.

4: Consider the product of the numbers on all the sheets after each operation.

5: Given B's choice (i, a_i), A should choose (j, a_j) so that $a_i 10^i + a_j 10^j$ does not depend on a_i.

6: Prove first that B can always make a move after A. Then consider what each person has to pick to end in a draw and see if B can prevent it.

7: Suppose A removes two equal numbers each round (prove first that this is possible). The game ends when the numbers left are distinct powers of 2 and 0's. Each 0 secretly comes from a power of 2. So A's minimal pay out is the number of powers of 2 needed to sum to 2020. Consider now B's strategy. Let $N(x_1, \ldots, x_k)$ be the number of choices for signs such that $\pm x_1 \pm \cdots \pm x_k = 0$. Then

$$\begin{aligned}
N(1, 1, \ldots, 1) &= \binom{2020}{1010} = 2^7 \cdot (\text{other prime factors}) \\
N(x_1, \ldots, x_k) &= N(x_1 + x_2, x_3, \ldots, x_k) + N(|x_1 - x_2|, x_3, \ldots, x_k) \\
N(\text{one number is the largest}) &= 0 \\
N(k\ \ 0's) &= 2^k
\end{aligned}$$

So B can't lose more than 7 by making sure that $2^8 \nmid N$ always.

8: Find a recursion formula relating the probability that each of them wins and solve it. End up comparing $\text{Re}(z_n)$, $\text{Re}(\zeta_3 z_n)$ and $\text{Re}(\zeta_3^2 z_n)$ where $z_n = \text{cis}(\theta_n)$ and $\theta_n = \sum_{k=1}^{n} \arctan \frac{\sqrt{3}}{2k-1}$.