Week 7

Number Theory

1. How many positive integers $x < 10^{2026}$ satisfy $10^{2026} \mid x^2 - x$.

2. Find all positive integer pairs (x, y) such that $1 + 2^x + 2^{2x+1} = y^2$.

3. Let a, b be positive integers such that $15a+16b$ and $16a-15b$ are both positive perfect squares. Find the smallest possible value for $\min\{15a + 16b, 16a - 15b\}$.

4. Let p be an odd prime. Define for any integer a,

$$S(a) = \sum_{j=1}^{p-1} \frac{a^j}{j}.$$

   Prove that when $S(3) + S(4) - 3S(2)$ is written in reduced form m/n, we have $p \mid m$.

5. For any odd positive integer q, define

$$N(q) = \#\{a \in \mathbb{Z} \colon 1 \leq a < q/4, \gcd(a, q) = 1\}.$$

   Prove that $N(q)$ is odd if and only if $q = p^k$ for some prime $p \equiv 5, 7 \pmod 8$ and positive integer k.

6. A subgroup G of $\mathbb{F}_p^\times$ is sum-free if there does not exist distinct elements $a, b, c \in G$ such that $a + b = c$. Let n be a positive integer. Prove that there exists a prime p such that $\mathbb{F}_p^\times$ has a sum-free subgroup of order n if and only if $6 \nmid n$.