Week 6

Integrals and derivatives

1. Find all twice continuously differentiable $f : \mathbb{R} \to \mathbb{R}$ such that $f(7x + 1) = 49f(x)$.

2. Determine real number c and positive L such that

$$\lim_{n\to\infty} n^c \frac{\displaystyle\int_0^{\pi/2} x^n \sin x \, dx}{\displaystyle\int_0^{\pi/2} x^n \cos x \, dx} = L.$$

3. Determine all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that for all $a, b > 0$,

$$f(b) - f(a) = (b - a)f'(\sqrt{ab}).$$

4. Suppose $f : (0, \infty) \to \mathbb{R}$ is continuous and decreasing with $f(x) \to 0$ as $x \to \infty$. Prove that

$$\int_0^\infty \frac{f(x) - f(x + 1)}{f(x)} dx \to \infty.$$

5. Does there exist a strictly increasing differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for every $x \in \mathbb{R}$.

6. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous with

$$\iint_R f(x, y) \, dxdy = 0$$

for all rectangles R of area 1. Prove that $f = 0$.

Integrals and derivatives

1. (IMC 2023 P1) Find all twice continuously differentiable $f : \mathbb{R} \to \mathbb{R}$ such that $f(7x+1) = 49f(x)$.

   Differentiate twice gives $f''(7x+1) = f''(x)$. Let $g(x) = f''(x)$. Then g is continuous. The map $T : x \mapsto (x-1)/7$ is a contraction on $\mathbb{R}$. So starting from any $x_0 \in \mathbb{R}$, the sequence $T^n x_0$ converges to the unique fixed point $x = -1/6$. Continuity then gives

   $$g(-1/6) = \lim_{n\to\infty} g(T^n x_0) = g(x_0).$$

   Hence $g(x)$ is a constant. So $f(x) = ax^2 + bx + c$. Setting $f(7x+1) = 49f(x)$ and solving for a, b, c gives $a = 3b$ and $b = 12c$. Hence

   $$f(x) = c(36x^2 + 12x + 1) = c(6x+1)^2.$$

2. (Putnam 2011 A3) Determine real number c and positive L such that

   $$\lim_{n\to\infty} n^c \frac{\displaystyle\int_0^{\pi/2} x^n \sin x\, dx}{\displaystyle\int_0^{\pi/2} x^n \cos x\, dx} = L.$$

   Define

   $$F(n) = \int_0^{\pi/2} x^n \sin x\, dx \qquad \text{and} \qquad G(n) = \int_0^{\pi/2} x^n \cos x\, dx.$$

   Using the bound $(2/\pi)x \le \sin x \le 1$, we have

   $$\frac{1}{n+2}\left(\frac{\pi}{2}\right)^{n+1} \le F(n) \le \frac{1}{n+1}\left(\frac{\pi}{2}\right)^{n+1}.$$

   From this, we see that

   $$\frac{\pi}{2}\frac{n+1}{n+3} \le \frac{F(n+1)}{F(n)} \le \frac{\pi}{2}.$$

   By integration by parts,

   $$G(n) = \frac{x^{n+1}\cos x}{n+1}\bigg|_0^{\pi/2} + \frac{1}{n+1}\int_0^{\pi/2} x^{n+1}\sin x\, dx = \frac{F(n+1)}{n+1}.$$

   Hence we have

   $$\frac{F(n)}{G(n)} = (n+1)\frac{F(n)}{F(n+1)}.$$

   Hence we see $c = -1$ and $L = 2/\pi$.

3. (IMC 2018 P4) Determine all differentiable functions $f : (0,\infty) \to \mathbb{R}$ such that for all $a, b > 0$,

   $$f(b) - f(a) = (b-a)f'(\sqrt{ab}).$$

First note that f' is differentiable: for any $t \in (0,\infty)$, we have

$$f'(t) = \frac{f(2t) - f(t/2)}{2t - t/2}.$$

We can now differentiate the original equation with respect to b to obtain

$$f'(b) = f'(\sqrt{ab}) + (b-a)\frac{\sqrt{a}}{2\sqrt{b}} f''(\sqrt{ab}).$$

Taking $a = 1/b$ gives

$$f'(b) = f'(1) + \frac{1}{2} f''(1) - \frac{1}{2b^2} f''(1).$$

Integrating gives

$$f(x) = Ax + B + Cx^{-1} \qquad \text{where} \qquad A, B, C \in \mathbb{R}.$$

We can now check that

$$f(b) - f(a) = A(b-a) + \frac{C(a-b)}{ab} = (b-a)\left(A - \frac{C}{ab}\right) = (b-a) f'(\sqrt{ab}).$$

4. (Putnam 2010 A6) Suppose $f : (0,\infty) \to \mathbb{R}$ is continuous and decreasing with $f(x) \to 0$ as $x \to \infty$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} dx \to \infty.$$

Suppose $n \le x \le n+1$. Then

$$1 - \frac{f(x+1)}{f(x)} + 1 - \frac{f(x+2)}{f(x+1)} \ge 1 - \frac{f(x+2)}{f(x)} \ge 1 - \frac{f(n+2)}{f(n+1)}$$

using $1 - a + 1 - b \ge 1 - ab$ for $a, b \le 1$. Let $g(x) = 1 - f(x+2)/f(x+1)$. Then

$$\prod_{n=1}^{\infty} (1 - g(n)) = \prod_{n=1}^{\infty} \frac{f(n+2)}{f(n+1)} \to 0.$$

Since $0 < g(n) < 1$, this is equivalent to

$$\sum_{n=1}^{\infty} \left(\int_n^{n+1} 1 - \frac{f(x+1)}{f(x)} dx + \int_{n+1}^{n+2} 1 - \frac{f(x+1)}{f(x)} dx \right) \ge \sum_{n=1}^{\infty} g(n) \to \infty.$$

5. (Putnam 2010 B5) Does there exist a strictly increasing differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for every $x \in \mathbb{R}$.

Since f is increasing, we have that f' is always positive and increasing. This implies that $f(\infty) = \infty$ and also $f'(\infty) = \infty$. In particular, for x large enough, we have $f(x) > x$. Now differentiate $f'(x) = f(f(x))$ to obtain

$$f''(x) = f'(f(x)) f'(x) > f'(x)^2 \qquad \text{for large enough } x.$$

The intuition now is that the solution to $y' = y^2$ is $y = -(x+c)^{-1}$ so we should have a contradiction with $f' > 0$. Differentiating $1/f'$ gives $-f''/(f')^2 < -1$. Hence $1/f'$ will eventually become negative. Contradiction.

6. (Putnam 2012 A6) Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous with

$$\iint_R f(x, y)\, dxdy = 0$$

for all rectangles R of area 1. Prove that $f = 0$.

Consider the rectangle $[a, a + c] \times [b, b + 1/c]$. Then

$$\int_a^{a+c} \int_b^{b+1/c} f(x, y)\, dydx = 0.$$

Differentiating with respect to a gives

$$\int_b^{b+1/c} f(a + c, y)dy = \int_b^{b+1/c} f(a, y)dy.$$

This shows that the two vertical edges have the same average. Differentiating with respect to b shows that the two horizontal edges have the same average. Differentiating with respect to c gives

$$\int_b^{b+1/c} f(a + c, y)dy + \int_a^{a+c} -\frac{1}{c^2} f(x, b + 1/c)dx = 0.$$

In other words,

$$c \int_b^{b+1/c} f(a + c, y)dy = \frac{1}{c} \int_a^{a+c} f(x, b + 1/c)dx,$$

so the average over the horizontal edge equals the average over the vertical edge.

Comparing the rectangles $[a, a + c] \times [b, b + 1/c]$ and $[a + c, a + 2c] \times [b, b + 1/c]$, since they share a vertical edge, gives

$$\int_a^{a+c} f(x, b)dx = \int_{a+c}^{a+2c} f(x, b)dx = \cdots = \int_{a+nc}^{a+(n+1)c} f(x, b)dx.$$

Hence for any a, b, d, n, we have

$$\frac{n}{d} \int_a^{a+d/n} f(x, b)dx = \frac{n}{d} \int_{a+d}^{a+d+d/n} f(x, b)dx.$$

Letting $n \to \infty$ gives $f(a, b) = f(a + d, b)$. Hence f is constant horizontally. The same argument shows f is constant vertically.