Week 5

Sequences

1. Let $a_1 = \sqrt{5}$, and let $a_n = a_{n-1}^2 - 2$ for $n \geq 2$. Determine whether $\lim_{n\to\infty} \dfrac{a_1 \cdots a_{n-1}}{a_n}$ exists and find its value if so.

2. Let $a_0 > 1$ be an integer. Define for $n \geq 0$,

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } a_n \text{ is a square,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

   Show that the sequence is eventually periodic if and only if $3 \mid a_0$.

3. Let $a_0 = a_1 = a_2 = 1$. Define for $n \geq 0$,

$$a_{n+3} = \frac{a_{n+1}a_{n+2} + n!}{a_n},$$

   where $0! = 1$. Is every term an integer?

4. Let $a_1 < a_2 < a_3 < \cdots$ be positive integers such that the sequences $(a_{a_n})_{n=1}^{\infty}$ and $(a_{a_n+1})_{n=1}^{\infty}$ are both arithmetic progressions. Show that $(a_n)_{n=1}^{\infty}$ is an arithmetic progression.

5. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers. Suppose there exists $k \geq 1$ such that

$$a_n = \max_{1\leq j\leq n} \left(a_j + a_{n-j}\right) \text{ for any } n > k.$$

   Show that there exists $\ell \geq 1$ such that $a_n = a_\ell + a_{n-\ell}$ for all n large enough.

6. Suppose $(a_n)_{n=1}^{\infty}$ is a sequence of integers such that $1 \leq a_n \leq 69$ for all n. Suppose that the numbers $n + a_n$ are all distinct. For any $r \in \mathbb{Z}$, let $s_r(n) = (a_1 - r) + \cdots + (a_n - r)$. Show that there exists $r, N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$|s_r(n) - s_r(m)| \leq 34^2.$$

Sequences

1. (IMC 2010 P3) Let $a_1 = \sqrt{5}$, and let $a_n = a_{n-1}^2 - 2$ for $n \geq 2$. Determine whether $\lim_{n\to\infty} \frac{a_1 \cdots a_{n-1}}{a_n}$ exists and find its value if so.

   We compute $a_2 = 3$, $a_3 = 7$, $a_4 = 47$. Note that

   $$a_n^2 = a_{n-1}^4 - 4a_{n-1}^2 + 4 = a_{n-1}^2(a_{n-1}^2 - 4) + 4.$$

   Hence

   $$a_n^2 - 4 = a_{n-1}^2 \cdots a_1^2(a_1^2 - 4) = (a_{n-1} \cdots a_1)^2.$$

   So the desired limit is

   $$\lim_{n\to\infty} \sqrt{1 - \frac{4}{a_n^2}} = 1$$

   provided we show that $a_n \to \infty$.

   We have

   $$a_n - a_{n-1} = (a_{n-1} + 1)(a_{n-1} - 2) \geq 4$$

   for $n \geq 3$. So $a_n \to \infty$.

   In fact, let $\varphi = \frac{1}{2}(\sqrt{5} + 1)$ so that $a_1 = \varphi + \varphi^{-1}$. It then follows by induction that

   $$a_{n+1} = \varphi^{2^n} + \varphi^{-2^n} \to \infty.$$

2. (IMO 2017 P1) Let $a_0 > 1$ be an integer. Define for $n \geq 0$,

   $$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } a_n \text{ is a square,} \\ a_n + 3 & \text{otherwise.} \end{cases}$$

   Show that the sequence is eventually periodic if and only if $3 \mid a_0$.

   First we note that any integer congruent to 2 mod 3 is not a square. So if $a_n \equiv 2 \pmod 3$ for any n, then all later terms are congruent to 2 mod 3 and we are simply adding 3 every time.

   If $a_n \equiv 1 \pmod 3$, then all future terms are 1 or 2 mod 3, and if 2 appears, then it goes off to infinity. We claim that it is impossible to have all terms being 1 mod 3. Suppose this is the case. Let m be the smallest term in the sequence. Note that $(m-2)^2 \geq m$ and is congruent to 1 mod 3. Hence starting from m, keep adding 3, the first square that appears is at most $(m-2)^2$. Then taking a squareroot produces a term at most $m-2$, contradicting minimality of m.

   Finally suppose $3 \mid a_0$. Then all terms are divisible by 3. Similarly, let m be the smallest term in the sequence. If $m \geq 6$, then $(m-3)^2 \geq m$ and the same argument as above gives a smaller term than m. Hence $m = 3$ and from that point on, the sequence repeats $3, 6, 9$.

3. (Putnam 2004 A3) Let $a_0 = a_1 = a_2 = 1$. Define for $n \geq 0$,

   $$a_{n+3} = \frac{a_{n+1}a_{n+2} + n!}{a_n},$$

where $0! = 1$. Is every term an integer?

Let $r_n = a_{n+2}/a_n$. Then we have

$$n! = \begin{vmatrix} a_n & a_{n+1} \\ a_{n+2} & a_{n+3} \end{vmatrix} = \begin{vmatrix} a_n & a_{n+1} \\ 0 & a_{n+3} - r_n a_{n+1} \end{vmatrix} = a_n a_{n+1}(r_{n+1} - r_n).$$

Divide this from

$$(n+1)! = a_{n+1}a_{n+2}(r_{n+2} - r_{n+1})$$

gives

$$n + 1 = r_n \frac{r_{n+2} - r_{n+1}}{r_{n+1} - r_n}.$$

Since $a_3 = 2$, we have $r_0 = 1$ and $r_1 = 2$. It is then easy to induct to get $r_n = n + 1$, so $a_{n+2} = (n + 1)a_n$ is an integer if a_n is. We are done by induction.

4. (IMO 2009 P3) Let $a_1 < a_2 < a_3 < \cdots$ be positive integers such that the sequences $(a_{a_n})_{n=1}^{\infty}$ and $(a_{a_n+1})_{n=1}^{\infty}$ are both arithmetic progressions. Show that $(a_n)_{n=1}^{\infty}$ is an arithmetic progression.

Let $b_n = a_{n+1} - a_n$. Let

$$D_1 = a_{a_{n+1}} - a_{a_n} \qquad \text{and} \qquad D_2 = a_{a_{n+1}+1} - a_{a_n+1}.$$

Note that

$$D_1 = \sum_{j=a_n}^{a_{n+1}-1} b_j$$

is a sum of b_n terms of the form b_j. Hence the sequence (b_n) is bounded. Let

$$m = \min\{b_n\} = b_r \qquad \text{and} \qquad M = \max\{b_n\} = b_s.$$

Then we have

$$D_1 = \sum_{j=a_r}^{a_{r+1}-1} b_j \le b_r M = mM \qquad \text{and} \qquad D_1 = \sum_{j=a_s}^{a_{s+1}-1} b_j \ge b_s m = mM.$$

Hence we have

$$b_{a_r} = \cdots = b_{a_{r+1}-1} = M \qquad \text{and} \qquad b_{a_s} = \cdots = b_{a_{s+1}-1} = m.$$

Now

$$D_2 - D_1 = b_{a_{n+1}} - b_{a_n}$$

is a constant. The boundedness of (b_n) implies that $b_{a_n} = c$ is a constant for all n. But then this gives $M = b_{a_r} = c = b_{a_s} = m$. Hence all of $b_j = c$.

5. (IMO 2010 P6) Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers. Suppose there exists $k \ge 1$ such that

$$a_n = \max_{1 \le j \le n} \left(a_j + a_{n-j}\right) \text{ for any } n > k.$$

Show that there exists $\ell \ge 1$ such that $a_n = a_\ell + a_{n-\ell}$ for all n large enough.

The intuition is that we expect to have $a_n = \frac{n}{\ell} a_\ell + O(1)$. Let ℓ be such that

$$\lambda = \max_{1 \le j \le k} \frac{a_j}{j} = \frac{a_\ell}{\ell}.$$

Let $b_n = \lambda n - a_n$. Then $b_1, \dots, b_k \ge 0$ with $b_\ell = 0$ and

$$b_n = \min_{1 \le j \le n} \big(b_j + b_{n-j} \big) \text{ for any } n > k.$$

From this, we see that each

$$b_n \in \{ c_1 b_1 + \cdots + c_k b_k \colon c_1, \dots, c_k \in \mathbb{Z}_{\ge 0} \}.$$

Moreover, if $n > k$, then

$$b_n \le b_\ell + b_{n-\ell} \le b_{n-\ell} \le \cdots \le \max\{b_1, \dots, b_k\} =: B.$$

It is easy to see that the set

$$\{ c_1 b_1 + \cdots + c_k b_k \colon c_1, \dots, c_k \in \mathbb{Z}_{\ge 0} \} \cap [0, B]$$

is finite. Hence for any $n > k$, the non-increasing sequence $b_n, b_{n+\ell}, b_{n+2\ell}, \dots$ is eventually constant. There are only finitely many residues mod ℓ. Hence we see that $b_n = b_{n-\ell}$ for all n large enough, which then gives

$$a_n = \lambda n - b_n = \lambda \ell + \lambda (n - \ell) - b_{n-\ell} = a_\ell + a_{n-\ell}.$$

6. (IMO 2015 P6) Suppose $(a_n)_{n=1}^\infty$ is a sequence of integers such that $1 \le a_n \le 69$ for all n. Suppose that the numbers $n + a_n$ are all distinct. For any $r \in \mathbb{Z}$, let $s_r(n) = (a_1 - r) + \cdots + (a_n - r)$. Show that there exists $r, N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$|s_r(n) - s_r(m)| \le 34^2.$$

Consider

$$\sum_{j=1}^n a_j = \sum_{j=1}^n (j + a_j) - \sum_{j=1}^n j.$$

Both sums are over distinct integers. We write

$$\sum_{j=1}^n a_j = \sum_{x \in E_n} x - \sum_{y \in R_n} y$$

where

$$\begin{aligned} E_n &= \{ j + a_j \colon j + a_j > n, j = 1, \dots, n \}, \\ R_n &= \{1, \dots, n\} \backslash \{ j + a_j \colon j + a_j \le n, j = 1, \dots, n \} \end{aligned}$$

denote the terms that are extra in the first sum and the second sum, respectively. The two sets have the same size and for $n > 69$, we have

$$E_n \subset \{ n + 1, \dots, n + 69 \}.$$

On the other hand, we have $R_{n+1} = R_n$ if $n+1 = j + a_j$ for some $j = 1, \ldots, n$; and $R_{n+1} = R_n \cup \{a_{n+1}\}$ otherwise. Hence there exists N such that for any $n > N$, we have $R_n = R_{n+1} =: R$. Let $r = |R_n| = |E_n|$ for all $n > N$. We have

$$s_r(n) = \sum_{j=1}^{n} (a_j - r) = \sum_{j=1}^{n} a_j - nr = \sum_{x \in E_n} (x - n) - \sum_{y \in R} y.$$

Let

$$F_n = E_n - n = \{x - n \colon x \in E_n\} \subset \{1, \ldots, 69\}.$$

Then for $n > m > N$, we have

$$s_r(n) - s_r(m) = \sum_{x \in F_n} x - \sum_{x \in F_m} x.$$

Since $R_n = R_{n+1}$, we have $n + 1 = j + a_j$ for some $j = 1, \ldots, n$ and so $n + 1 \in E_n$, implying that $1 \in F_n$. Similarly $1 \in F_m$. Hence the largest difference between the sum over F_n and F_m is achieved when F_n contains the $r - 1$ largest elements of $\{2, \ldots, 69\}$ and F_m contains the $r - 1$ smallest elements of $\{2, \ldots, 69\}$ (or vice-versa). Hence

$$|s_r(n) - s_r(m)| \leq (r - 1)(69 - r) \leq 34^2.$$