

Week 3

Probability

1. You have coins $C_1, C_2, \dots, C_n$. For each k , C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n .
2. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?
3. A sequence $y_1, y_2, \dots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \dots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \dots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \dots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \dots, i_k$ such that $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \dots, X_n)$ for $n \geq 2$.
4. Suppose X is a random variable that takes on only nonnegative integer values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here $E[Y]$ denotes the expectation of the random variable Y .) Determine the smallest possible value of the probability of the event $X = 0$.
5. Let A be a set of 2025 distinct positive integers. Show that there exists a subset $B \subset A$ of size 676 such that there does not exist $a, b, c \in B$ such that $a + b = c$.
6. Choose 2026 points independently and uniformly from the unit disk. Is the convex hull of these points more likely to have three vertices or four vertices?

Probability

1. (Putnam 2001 A2) You have coins $C_1, C_2, \dots, C_n$. For each k , C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n .

Strategy: Since only even/odd matters, we can encode it by the sign $(-1)^H$. Independence allows us to convert the sign of a sum into a product of expectations, and the resulting factors telescope.

Solution 1: Let I_k be 1 if coin C_k lands heads and 0 if it lands tails, and let $H = I_1 + \dots + I_n$. Then

$$(-1)^H = \prod_{k=1}^n (-1)^{I_k} \quad (1)$$

where the factors $(-1)^{I_k}$ are independent and

$$\mathbb{E}[(-1)^{I_k}] = 1 \cdot \frac{2k}{2k+1} + (-1) \cdot \frac{1}{2k+1} = \frac{2k-1}{2k+1}. \quad (2)$$

Therefore

$$\mathbb{E}[(-1)^H] = \prod_{k=1}^n \mathbb{E}[(-1)^{I_k}] = \prod_{k=1}^n \frac{2k-1}{2k+1} = \frac{1}{2n+1}. \quad (3)$$

Finally, since

$$\mathbb{E}[(-1)^H] = \mathbb{P}(H \text{ is even}) - \mathbb{P}(H \text{ is odd}) = 1 - 2\mathbb{P}(H \text{ is odd}), \quad (4)$$

we have $\mathbb{P}(H \text{ is odd}) = n/(2n+1)$.

Solution 2: Let p_n be the probability that the number of heads is odd after n tosses. Then

$$p_n = \frac{2n}{2n+1}p_{n-1} + \frac{1}{2n+1}(1 - p_{n-1}).$$

Multiplying by $2n+1$ gives

$$(2n+1)p_n = (2n-1)p_{n-1} + 1 = \dots = 3p_1 + n - 1 = n.$$

Hence

$$p_n = \frac{n}{2n+1}.$$

2. (Putnam 2002 B1) Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Strategy: Let H_r be the number of hits among the first r shots. Computing the distributions of H_r for small values of r suggests it's uniformly distributed. The rest is an induction argument.

Solution: We claim that, for every $r \geq 3$, the random variable H_r is uniform on $\{1, \dots, r-1\}$. When $r = 3$, she has $1/2$ chance of miss so $H_3 = 1$ and $1/2$ chance of hit so $H_3 = 2$.

Suppose now $r \geq 4$. Fix $j = 1, \dots, r-1$. Then

$$\Pr(H_r = j) = \Pr(H_{r-1} = j-1) \frac{j-1}{r-1} + \Pr(H_{r-1} = j) \frac{r-1-j}{r-1} = \frac{1}{r-1}.$$

Hence $\Pr(H_{100} = 50) = 1/99$.

3. (Putnam 2023 B3) A sequence $y_1, y_2, \dots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \dots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \dots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \dots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1, i_2, \dots, i_k$ such that $X_{i_1}, X_{i_2}, \dots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \dots, X_n)$ for $n \geq 2$.

Strategy: Keep the two endpoints and every local maximum and minimum; this gives a longest zigzag. Then count the expected number of local extrema.

Solution: With probability 1, the numbers $X_1, \dots, X_n$ are distinct. For each $2 \leq i \leq n-1$, let I_i be 1 if X_i is a local max or local min and 0 otherwise. That is, $I_i = 1$ if X_i is the biggest or the smallest among X_{i-1}, X_i, X_{i+1} . Hence we have $E(I_i) = 2/3$.

Next we observe that

$$a(X_1, \dots, X_n) = 1 + I_2 + \dots + I_{n-1} + 1.$$

By taking the X_i where $I_i = 1$ and the two end end points, we get a zigzag of length $2 + I_2 + \dots + I_{n-1}$. Conversely, any zigzag y_1, y_2, y_3 forces a local min and local max between y_1 and y_3 . Hence the total number of local extrema is at least $a(X_1, \dots, X_n) - 2$.

Linearity of expectation gives

$$E(a(X_1, \dots, X_n)) = 2 + (n-2) \frac{2}{3} = \frac{2n+2}{3}.$$

4. (Putnam 2014 A4) Suppose X is a random variable that takes on only nonnegative integer values, with $E[X] = 1$, $E[X^2] = 2$, and $E[X^3] = 5$. (Here $E[Y]$ denotes the expectation of the random variable Y .) Determine the smallest possible value of the probability of the event $X = 0$.

Strategy: Write $p_j = \Pr(X = j)$. Because the moments are known through degree three, we can compute the expectation of any cubic.

Solution: The random variable $(X-1)(X-2)(X-3)$ equals -6 when $X = 0$, equals 0 when $X = 1, 2, 3$, and is positive when $X \geq 4$. Its expectation is

$$E[(X-1)(X-2)(X-3)] = E(X^3) - 6E(X^2) + 11E(X) - 6 = -2.$$

Hence we have

$$-2 = -6p_0 + \sum_{j \geq 4} (j-1)(j-2)(j-3)p_j \geq -6p_0.$$

Hence we find that $p_0 \geq 1/3$.

To how $p_0 = 1/3$ can be achieved, we note that there exist p_1, p_2, p_3 satisfying the Vandermonde system

$$\begin{aligned} p_1 + p_2 + p_3 &= 1 - p_0, \\ p_1 + 2p_2 + 3p_3 &= E(X) = 1, \\ p_1 + 4p_2 + 9p_3 &= E(X^2) = 2. \end{aligned}$$

Then $E[(X-1)(X-2)(X-3)] = -6p_0 = -2$ which forces $E(X^3) = 5$.

5. (2001 USA TST P3) Let A be a set of 2025 distinct positive integers. Show that there exists a subset $B \subset A$ of size 676 such that there does not exist $a, b, c \in B$ such that $a + b = c$.

Solution: Choose $t \in (0, 1)$ uniformly at random. Let

$$B_t = \{a \in A: \frac{1}{3} < \{ta\} < \frac{2}{3}\}.$$

For any $a, b \in B_t$, we have $ta + tb - \lfloor ta \rfloor - \lfloor tb \rfloor \in (2/3, 4/3)$ and so $\{t(a+b)\} \in (2/3, 1) \cup [0, 1/3)$. This shows that B_t is sum-free. Linearity of expectation gives

$$E(\#B_t) = \sum_{a \in A} \Pr(1/3 < \{ta\} < 2/3) = 675.$$

Let $M = \max\{a \in A\}$. Then for $t \in (0, 1/3M)$, we see that $B_t = \emptyset$. Hence there is a positive probability for $\#B_t < 675$, which means there is positive probability for $\#B_t > 675$.

6. (2019 IMC P10) Choose 2026 points independently and uniformly from the unit disk. Is the convex hull of these points more likely to have three vertices or four vertices?

Strategy: The key observation here is to notice that choosing 4 points has way more possible combinations than 3, more precisely

$$\frac{\binom{2026}{4}}{\binom{2026}{3}} = \frac{2023}{4}. \quad (5)$$

At this point, we can speculate that it's much more likely that the answer has 4 vertices, and we only need a relatively crude bound to show the probability is strictly greater.

Solution: Let $X_1, \dots, X_{2026}$ be the chosen points. Let a be the probability that the convex hull is formed from X_1, X_2, X_3 and let b be the probability that the convex hull is formed from X_1, X_2, X_3, X_4 . Then

$$\Pr(\text{triangular hull}) = \binom{2026}{3} a, \quad \Pr(\text{quadrilateral hull}) = \binom{2026}{4} b.$$

It remains to prove that $a < (2023/4)b$. We show in fact $b > a/3$.

Fix A, B, C in the disk. Let T be the triangle ABC . Let $G = (A + B + C)/3$ be the centroid. Consider the reflections

$$A' = 2G - A, \quad B' = 2G - B, \quad C' = 2G - C.$$

We show first that at least one of them still lie inside the circle. To show this, let O denote the center of the disk. Then

$$\|A' - O\|^2 + \|B' - O\|^2 + \|C' - O\|^2 = \|A - O\|^2 + \|B - O\|^2 + \|C - O\|^2 \leq 3r^2.$$

To this this, we translate so that $G = 0$. Then $A + B + C = 0$. So

$$\begin{aligned} & \|A - O\|^2 + \|B - O\|^2 + \|C - O\|^2 \\ &= \|A\|^2 + \|B\|^2 + \|C\|^2 + 3\|O\|^2 - 2\langle O, A + B + C \rangle \\ &= \|A\|^2 + \|B\|^2 + \|C\|^2 + 3\|O\|^2 \end{aligned}$$

which doesn't change when A, B, C are replaced by $-A, -B, -C$.

Suppose A' is in the disk. Let $D = (B + C)/2$ be the midpoint of B, C . Then

$$G = \frac{1}{3}A + \frac{2}{3}D = D - \frac{1}{3}(D - A)$$

is on the line AD and $1/3$ of the way to D . Then

$$A' = 2G - A = D + \frac{1}{3}(D - A)$$

is on the line AD and $1/3$ of the way past D . This shows that

$$\text{Area}(BCA') = \text{Area}(BCG) = \frac{1}{3}\text{Area}(ABC).$$

Suppose now $X_1 = A, X_2 = B, X_3 = C$. The only way for X_1, X_2, X_3 to be the convex hull is if all of $X_4, \dots, X_{2026}$ lie inside ABC . So

$$a|(X_1 = A, X_2 = B, X_3 = C) = \left(\frac{\text{Area}(ABC)}{\pi} \right)^{2023}.$$

If X_4 lies in the triangle BCA' and $X_5, \dots, X_{2026}$ lie inside ABC . then we have X_1, X_2, X_3, X_4 as the convex hull. Hence

$$b|(X_1 = A, X_2 = B, X_3 = C) \geq \frac{\text{Area}(A'BC)}{\pi} \cdot \left(\frac{\text{Area}(ABC)}{\pi} \right)^{2022} \geq \frac{1}{3}a.$$

Extra: Consider still the triangle $T = ABC$. Extend AB, AC, BC out to intersect the disk D . The disk is now cut up into 7 pieces, the triangle T , three triangular fans, and three quadrilateral fans R_1, R_2, R_3 . One can ask: what is the largest c such that

$$\text{Area}(R_1) + \text{Area}(R_2) + \text{Area}(R_3) \geq c \cdot \text{Area}(T).$$

The answer is when T is an equilateral triangle inscribed on the circle in which case

$$c = \frac{4\pi}{3\sqrt{3}} - 1.$$

The key here is that there is an ellipse E inscribing T contained in D . Suppose D is the unit disk. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$. Since they are not colinear, there exist $a, b, c \in \mathbb{R}$ such that

$$\begin{aligned} a + bx_1 + cy_1 &= \sqrt{1 - x_1^2 - y_1^2}, \\ a + bx_2 + cy_2 &= \sqrt{1 - x_2^2 - y_2^2}, \\ a + bx_3 + cy_3 &= \sqrt{1 - x_3^2 - y_3^2}. \end{aligned}$$

Hence, A, B, C lie on the ellipse

$$E = \{(x, y) : x^2 + y^2 + (a + bx + cy)^2 = 1\}$$

which is clearly contained in D since $x^2 + y^2 \leq 1$ if $(x, y) \in E$. Since A, B, C lie on the boundary of E , the three triangular fans all lie outside of E . So

$$\text{Area}(R_1) + \text{Area}(R_2) + \text{Area}(R_3) \geq \text{Area}(E) - \text{Area}(T).$$

Dividing by $\text{Area}(T)$, we see that it remains to minimize $\text{Area}(E)/\text{Area}(T)$. The second key point is that this quotient is invariant under invertible affine linear change of variables since both areas would be scaled by the same Jacobian. We can then apply an affine linear change of variables to turn E into the unit disk D . The question now becomes the classical question of maximizing the area of an inscribed triangle inside a unit disk.