

Week 2

Combinatorics

1. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k -element subset $X \subset \{1, 2, \dots, k + a\}$ uniformly and, independently of X , choose a random n -element subset $Y \subset \{1, \dots, k + n + a\}$ uniformly. Prove that the probability that $\min(Y) > \max(X)$ does not depend on a .
2. How many words of length 2026 from the alphabet $\{a, b, c, d\}$ are there with an even number of a 's and b 's.
3. For $1 \leq j \leq i \leq n$, we set $a_{i,j}$ to be 0 or 1 such that $\sum_{j=1}^i a_{i,j} = 1$. Define recursively $b_{i,j} = \max\{b_{i+1,j}, b_{i+1,j+1}\} + a_{i,j}$ for $1 \leq j \leq i \leq n$, and $b_{n+1,j} = 0$ for all j . Prove that $b_{1,1} \geq \log_2(n+1)$.
4. A reverse Pascal triangle of size n consists of real numbers $a_{i,j}$ for $1 \leq j \leq i \leq n$ such that $a_{i,j} = |a_{i+1,j} - a_{i+1,j+1}|$ for $i \leq n-1$. Does there exist a reverse Pascal triangle of size 2026 consisting of integers from 1 to $1 + 2 + \dots + 2026$ (each exactly once)?
5. Let S_n denote the set of permutations on $\{1, \dots, n\}$. Let $k = 1, \dots, n$ and let $\mathcal{F} \subset S_n$ be a subset such that for any $\sigma \in \mathcal{F}$, there exists $1 \leq i_1 < \dots < i_k \leq n$ such that for any $j = 1, \dots, k-1$, the permutation $\sigma \cdot (i_j, i_{j+1})$ (i.e. obtained from σ by swapping $\sigma(i_j)$ with $\sigma(i_{j+1})$) is also in $\mathcal{F}$. Prove that $\#\mathcal{F} \geq k!$.
6. For any map $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, let $o(\sigma)$ denote the number of distinct elements in $\{\sigma, \sigma \circ \sigma, \sigma \circ \sigma \circ \sigma, \dots\}$. Let $\epsilon > 0$ be any real number. Prove that for n large enough, we have

$$\max_{\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}} o(\sigma) \leq \max_{\sigma \in S_n} o(\sigma) + n^{0.5+\epsilon}.$$

1. (IMC 2021 P2) Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k -element subset $X \subset \{1, 2, \dots, k+a\}$ uniformly and, independently of X , choose a random n -element subset $Y \subset \{1, \dots, k+n+a\}$ uniformly. Prove that the probability that $\min(Y) > \max(X)$ does not depend on a .

In order for $\min(Y) > \max(X)$, the union $X \cup Y$ is an $(k+n)$ -element subset $Z \subset \{1, \dots, k+n+a\}$. Conversely, such a Z uniquely determine X, Y with $\min(Y) > \max(X)$. Hence the desired probability is

$$\frac{\binom{k+n+a}{n+k}}{\binom{k+a}{k} \binom{k+n+a}{n}} = \binom{n+k}{n}^{-1}.$$

2. (IMC 2020 P1) How many words of length 2026 from the alphabet $\{a, b, c, d\}$ with an even number of a 's and b 's.

For $i, j = 0, 1$, let $N_{ij}(n)$ denote the number of words of length n where the number of a 's is $i \bmod 2$, and the number of b 's is $j \bmod 2$. Then we have

$$\begin{pmatrix} N_{00}(n) \\ N_{10}(n) \\ N_{11}(n) \\ N_{01}(n) \end{pmatrix} = A \begin{pmatrix} N_{00}(n-1) \\ N_{10}(n-1) \\ N_{11}(n-1) \\ N_{01}(n-1) \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

It is easy to observe that A has eigenvalue 4 with eigenvector $(1, 1, 1, 1)$; eigenvalue 2 with eigenvectors $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$; and finally eigenvalue 0 with eigenvector $(1, -1, 1, -1)$. Hence

$$\begin{aligned} \begin{pmatrix} N_{00}(n) \\ N_{10}(n) \\ N_{11}(n) \\ N_{01}(n) \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 4^{n-1} & & & \\ & 2^{n-1} & & \\ & & 2^{n-1} & \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 4^{n-1} & & & \\ & 2^{n-1} & & \\ & & 2^{n-1} & \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 4^{n-1} \\ 2^{n-1} \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Hence we see that $N_{00}(n) = 4^{n-1} + 2^{n-1}$.

3. (IMO 2023 P5) For $1 \leq j \leq i \leq n$, we set $a_{i,j}$ to be 0 or 1 such that $\sum_{j=1}^i a_{i,j} = 1$. Define recursively $b_{i,j} = \max\{b_{i+1,j}, b_{i+1,j+1}\} + a_{i,j}$ for $1 \leq j \leq i \leq n$, and $b_{n+1,j} = 0$ for all j . Prove that $b_{1,1} \geq \log_2(n+1)$.

Consider $S(i) = \sum_{j=1}^i 2^{b_{i,j}}$. Then $S(n) = n - 1 + 2 = n + 1$ and $S(1) = 2^{b_{1,1}}$. It suffices to show that $S(i) \geq S(i+1)$ for all $i = 1, \dots, n-1$. Fix i and suppose $b_{i+1,m}$ is minimal among all $b_{i+1,j}$ for $j = 1, \dots, i$. Then

$$2^{\max\{b_{i+1,j}, b_{i+1,j+1}\}} \geq 2^{b_{i+1,j}} \text{ for } j < m; \text{ and } 2^{\max\{b_{i+1,j}, b_{i+1,j+1}\}} \geq 2^{b_{i+1,j+1}} \text{ for } j \geq m.$$

Suppose $a_{i,j_0} = 1$. Then

$$S(i) = \sum_{j=1}^i 2^{b_{i,j}} \geq \sum_{j=1}^{i+1} 2^{b_{i+1,j}} - 2^{b_{i+1,m}} + (2^{b_{i+1,j_0}} \text{ or } 2^{b_{i+1,j_0+1}}) \geq S(i+1).$$

4. (IMO 2018 P3) A reverse Pascal triangle of size n consists of real numbers $a_{i,j}$ for $1 \leq j \leq i \leq n$ such that $a_{i,j} = |a_{i+1,j} - a_{i+1,j+1}|$ for $i \leq n-1$. Does there exist a reverse Pascal triangle of size 2026 consisting of integers from 1 to $1 + 2 + \dots + 2026$ (each exactly once)?

For each $i = 1, \dots, n$, let

$$m_i = \min_{1 \leq j \leq i} a_{i,j} \quad \text{and} \quad M_i = \max_{1 \leq j \leq i} a_{i,j}.$$

Then we see that $M_i \leq M_{i+1} - m_{i+1}$ so

$$1 + 2 + \dots + n \geq M_n \geq m_n + \dots + m_2 + M_1 = m_n + \dots + m_2 + m_1.$$

Hence $M_n = 1 + 2 + \dots + n$ and $m_1, \dots, m_n$ is a permutation of $1, 2, \dots, n$.

We say an entry (i, j) is large if $1 + 2 + \dots + (n-1) \leq a_{i,j} \leq 1 + 2 + \dots + n$. If $a_{i,j}$ is large, then one of $a_{i+1,j}$ or $a_{i+1,j+1}$ is at most n , so either $a_{i+1,j} = m_{i+1}$ or $a_{i+1,j+1} = m_{i+1}$. In other words, if $i < n$, then at most two entries (i, j) can be large.

Suppose now $1 \leq k < n$ such that $n \leq 1 + 2 + \dots + (n-k)$. Then

$$M_k = m_1 + \dots + m_k \leq n + (n-1) + \dots + (n-k+1) \leq 1 + 2 + \dots + (n-1).$$

So there are no large entries with $i \leq k$ and at most two for $i = k+1, \dots, n-1$. Hence there are at least $n - 2(n-k-1)$ large entries of the form (n, j) . If $a_{n,j}$ and $a_{n,j+1}$ are both large, then $a_{n-1,j} \leq n$ so $a_{n-1,j} = m_{n-1}$. Hence at most 2 of the large entries of the form (n, j) can be adjacent. So $n - 2(n-k-1) \leq 2 + \lceil (n-3)/2 \rceil$. Since $n-k$ can be taken to be $\lceil \sqrt{2n} \rceil$, we have

$$n - 2(\lceil \sqrt{2n} \rceil - 1) \leq 2 + \lceil (n-3)/2 \rceil$$

which does not hold when $n \geq 35$.

5. (IMC 2023 P5) Let S_n denote the set of permutations on $\{1, \dots, n\}$. Let $k = 1, \dots, n$ and let $\mathcal{F} \subset S_n$ be a subset such that for any $\sigma \in \mathcal{F}$, there exists $1 \leq i_1 < \dots < i_k \leq n$ such that for any $j = 1, \dots, k-1$, the permutation $\sigma \cdot (i_j, i_{j+1})$ (i.e. obtained from σ by swapping $\sigma(i_j)$ with $\sigma(i_{j+1})$) is also in $\mathcal{F}$. Prove that $\#\mathcal{F} \geq k!$.

Define a pairing $\langle, \rangle$ on S_n by $\langle \sigma, \tau \rangle = \sum_{i=1}^n \sigma(i)\tau(i)$. For any $\sigma \in \mathcal{F}$, let

$$A(\sigma) = \{\gamma \in S_n : \forall \tau \in \mathcal{F}, \langle \sigma, \gamma \rangle \geq \langle \tau, \gamma \rangle\}.$$

For every $\gamma \in S_n$, there exists $\sigma \in \mathcal{F}$ that maximizes $\langle \sigma, \gamma \rangle$ which implies that $\gamma \in A(\sigma)$. Hence, the union of $A(\sigma)$ as σ varies in $\mathcal{F}$ covers S_n . It remains to show that each $\#A(\sigma) \leq n!/k!$.

Fix $\sigma \in \mathcal{F}$ and $1 \leq i_1 < \dots < i_k \leq n$ such that for any $j = 1, \dots, k-1$, the permutation $\sigma_j = \sigma \cdot (i_j, i_{j+1}) \in \mathcal{F}$. Let $\gamma \in A(\sigma)$. Then $\langle \sigma, \gamma \rangle \geq \langle \sigma_j, \gamma \rangle$, which implies

$$\sigma(i_j)\gamma(i_j) + \sigma(i_{j+1})\gamma(i_{j+1}) \geq \sigma(i_{j+1})\gamma(i_j) + \sigma(i_j)\gamma(i_{j+1}).$$

Rearranging gives

$$(\sigma(i_{j+1}) - \sigma(i_j))(\gamma(i_{j+1}) - \gamma(i_j)) \geq 0.$$

In other words, $\gamma(i_1), \dots, \gamma(i_k)$ are in the same order as $\sigma(i_1), \dots, \sigma(i_k)$. This gives at most $n!/k!$ such permutations.

6. (IMC 2025 P5) For any map $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, let $o(\sigma)$ denote the number of distinct elements in $\{\sigma, \sigma \circ \sigma, \sigma \circ \sigma \circ \sigma, \dots\}$. Let $\epsilon > 0$ be any real number. Prove that for n large enough, we have

$$\max_{\sigma: \{1, \dots, n\} \rightarrow \{1, \dots, n\}} o(\sigma) \leq \max_{\sigma \in S_n} o(\sigma) + n^{0.5+\epsilon}.$$

Fix any $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. For any $i \in \{1, \dots, n\}$, the sequence $\sigma^n(i)$ is eventually periodic. Let

$$P(\sigma) = \{j \in \{1, \dots, n\} : \sigma^k(j) = j \text{ for some } k \geq 1\}$$

be the union of the periodic part of $\sigma^n(i)$ as i varies. For any $j_1 \neq j_2 \in P(\sigma)$, there exists k such that $\sigma^k(j_1) = j_1$ and $\sigma^k(j_2) = j_2$, and so $\sigma(j_1) \neq \sigma(j_2)$. It is then easy to see that $\sigma : P(\sigma) \rightarrow P(\sigma)$ is a permutation and

$$N(\sigma) := o(\sigma|_{P(\sigma)}) \leq \max_{\tau \in S_n} o(\tau).$$

Let $R(\sigma)$ be the smallest $r \geq 0$ such that for any $i = 1, \dots, n$, we have $\sigma^r(i) \in P(\sigma)$. That is, $R(\sigma)$ is the longest pre-periodic part of $\sigma^n(i)$. Then $\sigma^{R(\sigma)} = \sigma^{R(\sigma)+N(\sigma)}$, so $o(\sigma) < R(\sigma) + N(\sigma)$. We are done if $R(\sigma) \leq n^{0.5+\epsilon}$.

Suppose now $R(\sigma) > n^{0.5+\epsilon}$. Then since $P(\sigma)$ is missing at least $n^{0.5+\epsilon}$ elements, we want to find a prime $p < n^{0.5+\epsilon}$ that is coprime to $N(\sigma)$, so that by adding a p -cycle to $\sigma|_{P(\sigma)}$ using elements not in $P(\sigma)$, we obtain a permutation $\tau \in S_n$ with $o(\tau) = pN(\sigma)$ so that

$$o(\sigma) < n + \frac{1}{p} \max_{\tau \in S_n} o(\tau) \leq n + \frac{1}{2} \max_{\tau \in S_n} o(\tau) < \max_{\tau \in S_n} o(\tau)$$

for n large enough.

Recall that $N(\sigma)$ is the least common multiple of all the cycle lengths of $\sigma|_{P(\sigma)}$. Since the product of two positive integers is larger than their sum, we see that sum of the (distinct) prime divisors of the cycle lengths of $\sigma|_{P(\sigma)}$ is less than $\#P(\sigma) < n - n^{0.5+\epsilon}$. On the other hand, we know that

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} p \gg \frac{x^2}{\ln x}.$$

(In fact, it is asymptotic to $x^2/(2 \ln x)$.) Hence the sum of all primes up to $n^{0.5+\epsilon}$ is $\gg n^{1+2\epsilon}/\ln n$, which exceeds n for n large enough. Therefore, there exists a prime $p < n^{0.5+\epsilon}$ coprime to $N(\sigma)$.