

Week 1

Pigeonhole Principle

1. Let n be a positive integer and let $x_1, \dots, x_n \in \mathbb{R}$. Prove that there is a non-empty subset $I \subset \{1, 2, \dots, n\}$ such that the distance from $\sum_{i \in I} x_i$ to the nearest integer is at most $1/(n+1)$.
2. Find the smallest positive integer N such that given any N distinct positive integers, none of which is divisible by a prime larger than 23, there exists 4 distinct numbers that multiply to a perfect 4th power.
3. The set $S = \{1, 2, \dots, 2026\}$ is partitioned into 6 sets $A_1, A_2, \dots, A_6$. Prove that there exists $j = 1, 2, \dots, 6$, and $x, y, z \in A_j$ such that $x + y = z$.
4. Let $g \neq h$ be elements of a group G . For any $j = 1, 2, \dots$, let $w_j \in \{g, h\}$ such that the sequence $(w_n)_{n=1}^\infty$ is not eventually periodic. Let $n \geq 1$ be an integer. Let H be the subgroup of G generated by $w_k w_{k+1} \cdots w_{k+n-1}$ for all $k = 1, 2, \dots$. Let K be the subgroup of G generated by all length n words formed using g, h . Prove that $H = K$.
5. Let n be a positive integer and let $x_1, \dots, x_n \in \mathbb{R}$. For any $I \subset \{1, 2, \dots, n\}$, let $s(I) = \sum_{i \in I} x_i$. Suppose the set $\{s(I) : I \subset \{1, 2, \dots, n\}\}$ has at least 1.9^n (distinct) elements. Prove that the set $\{J \subset \{1, 2, \dots, n\} : s(J) = 69\}$ has at most 1.6^n (distinct) elements.
6. Let n be a positive integer and let $a_1, \dots, a_n$ be distinct positive real numbers. Let S be a set of at most $n - 1$ positive real numbers not containing $a_1 + \cdots + a_n$. Prove that there is a bijection $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that none of $a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(j)}$, for $j = 1, \dots, n$, belong to S .

1. (Putnam 2006 B2) Let n be a positive integer and let $x_1, \dots, x_n \in \mathbb{R}$. Prove that there is a non-empty subset $I \subset \{1, 2, \dots, n\}$ such that the distance from $\sum_{i \in I} x_i$ to the nearest integer is at most $1/(n+1)$.

Divide $[0, 1)$ into $n+1$ intervals $J_0, \dots, J_n$ with $J_k = [k/(n+1), (k+1)/(n+1))$ for $k = 0, \dots, n$. Consider the n real numbers $a_1, \dots, a_n$ defined by

$$a_k = \left\{ \sum_{i=1}^k x_i \right\}$$

where $\{ \}$ denote taking the fractional part. If any of the a_k 's belong to J_0 or J_n , then we are done. Otherwise, two of them belong to the same J_k for some k , and their difference does the job.

2. (IMO 1985 P4) Find the smallest positive integer N such that given any N distinct positive integers, none of which is divisible by a prime larger than 23, there exists 4 distinct numbers that multiply to a perfect 4th power.

There are 9 primes under 23. Express any positive integer with no prime divisors larger than 23 as $m = 2^{e_1} 3^{e_2} 5^{e_3} 7^{e_4} 11^{e_5} 13^{e_6} 17^{e_7} 19^{e_8} 23^{e_9}$ and map it to $e(m) = (e_1, \dots, e_9) \pmod{2} \in \mathbb{F}_2^9$. If $N \geq 513$, then we can find two such elements m_1, n_1 with $e(m_1) = e(n_1)$ so they multiply to a square. If $N - 2 \geq 513$, we can find another two such elements m_2, n_2 with $e(m_2) = e(n_2)$. We can find such pairs m_j, n_j for $j = 1, \dots, 513$ if $N - 2 \cdot 512 \geq 513$. Let $a_j = \sqrt{m_j n_j}$. Then a_j is positive integer with no prime divisors larger than 23. We can then find $j \neq k$ such that $e(a_j) = e(a_k)$. It then follows that $m_j n_j m_k n_k = (a_j a_k)^2$ is a fourth power. Hence we have shown that $N = 1537$ works.

To construct an example where $N = 1536$ fails, take $a_1, a_2, \dots, a_{512}$ so that their exponents run through all 512 possible 9-tuples of 0 and 1. Then we take

$$a_1, \dots, a_{512}, 2^4 a_1, \dots, 2^4 a_{512}, 3^4 a_1, \dots, 3^4 a_{512}.$$

Then it suffices to show that when we choose 4 numbers from $a_1, \dots, a_{512}$, at most 3 of which can be same, their product is not a fourth power. This follows because in order for the exponents to be divisible by 4, the four numbers must have the same exponents.

3. (IMO 1978 P6) The set $S = \{1, 2, \dots, 2026\}$ is partitioned into 6 sets $A_1, A_2, \dots, A_6$. Prove that there exists $j = 1, 2, \dots, 6$, and $x, y, z \in A_j$ such that $x + y = z$.

Suppose not. By Pigeonhole, we may assume that $|A_1| \geq 338$. Let $a_1 < \dots < a_{338}$ be in A_1 . The differences $a_2 - a_1, \dots, a_{338} - a_1$ are in S but not A_1 . By Pigeonhole, we may assume that A_2 contains at least 68 of them. Label them $b_1 < \dots < b_{68}$. Note that the difference of any of the two b_i 's is a difference of the a_i 's. The differences $b_2 - b_1, \dots, b_{68} - b_1$ all belong to S but not A_1 or A_2 . By Pigeonhole, we may assume A_3 contains at least 17 of them. Label them $c_1 < \dots < c_{17}$. Similarly, the difference of any of the two b_i 's is a difference of the a_i 's. The differences $c_2 - c_1, \dots, c_{17} - c_1$ all belong to S but not A_1 or A_2 or A_3 . By Pigeonhole, we may assume A_4 contains at least 6 of them, labeled $d_1 < \dots < d_6$. Similarly, the differences $d_2 - d_1, \dots, d_6 - d_1$ belong to S but not A_1, A_2, A_3, A_4 . By Pigeonhole, we may assume A_5 contains at least 3 of them $e_1 < e_2 < e_3$. Finally, the differences $e_2 - e_1$ and $e_3 - e_1$ both belong to A_6 , but their difference $e_3 - e_2$ does not belong to any of the six sets.

4. (IMC 2024 P4) Let $g \neq h$ be elements of a group G . For any $j = 1, 2, \dots$, let $w_j \in \{g, h\}$ such that the sequence $(w_n)_{n=1}^\infty$ is not eventually periodic. Let $n \geq 1$ be an integer. Let H be the subgroup of G generated by $w_k w_{k+1} \cdots w_{k+n-1}$ for all $k = 1, 2, \dots$. Let K be the subgroup of G generated by all length n words formed using g, h . Prove that $H = K$.

It is clear that $H \subset K$. By Pigeonhole, there exists positive integers $k \neq m$ such that the n -tuples

$$(w_k, w_{k+1}, \dots, w_{k+n-1}) = (w_m, w_{m+1}, \dots, w_{m+n-1}).$$

Since the sequence (w_n) is not eventually periodic, we may assume that $w_{k+n} = g \neq h = w_{m+n}$. Then

$$g^{-1}h = (w_{k+1} \cdots w_{k+n})^{-1}(w_{m+1} \cdots w_{m+n}) \in H$$

and similarly for $h^{-1}g$. Consider next

$$(w_{k+2} \cdots w_{k+n+1})^{-1}(w_{m+2} \cdots w_{m+n+1}) = w_{k+n+1}^{-1}g^{-1}hw_{m+n+1} \in H.$$

By multiplying on the right and left by $g^{-1}h$ or $h^{-1}g$, we see that H contains all elements of the form $a^{-1}b$ where a, b are length 2 words formed using g, h . It then follows by induction on $j = 2, \dots, n$ using $(w_{k+j} \cdots w_{k+j+n-1})^{-1}(w_{m+j} \cdots w_{m+j+n-1})$ that H contains all elements of the form $a^{-1}b$ where a, b are length j words formed using g, h . The statement for $j = n$ implies that $K \subset H$.

5. (IMC 2019 P8) Let n be a positive integer and let $x_1, \dots, x_n \in \mathbb{R}$. For any $I \subset \{1, 2, \dots, n\}$, let $s(I) = \sum_{i \in I} x_i$. Suppose the set $\{s(I) : I \subset \{1, 2, \dots, n\}\}$ has at least 1.9^n (distinct) elements. Prove that the set $\{J \subset \{1, 2, \dots, n\} : s(J) = 69\}$ has at most 1.6^n (distinct) elements.

Let X consist of at least 1.9^n subsets of $\{1, \dots, n\}$ such that $s(I)$ for $I \in X$ are all distinct. Let $Y = \{J \subset \{1, 2, \dots, n\} : s(J) = 69\}$ and suppose for contradiction that $|Y| > 1.6^n$. Given $I \in X$ and $J \in Y$, define $f_{I,J} : \{1, \dots, n\} \rightarrow \{-1, 0, 1\}$ by

$$f_{I,J}(a) = \begin{cases} 1 & \text{if } a \in I, a \notin J, \\ 0 & \text{if } a \in I, a \in J \text{ or } a \notin I, a \notin J, \\ -1 & \text{if } a \notin I, a \in J. \end{cases}$$

The point is that

$$\sum_{a=1}^n f_{I,J}(a)x_a = s(I) - s(J) = s(I) - 69.$$

Since $|X \times Y| > 1.9^n \cdot 1.6^n > 3^n$, by Pigeonhole, there exist distinct pairs $(I_1, J_1), (I_2, J_2) \in X \times Y$ such that $f_{I_1, J_1} = f_{I_2, J_2}$, which implies that $s(I_1) = s(I_2)$. Hence $I_1 = I_2$, which then implies that $J_1 = J_2$, contradiction.

6. (IMO 2009 P6) Let n be a positive integer and let $a_1, \dots, a_n$ be distinct positive real numbers. Let S be a set of at most $n - 1$ positive real numbers not containing $a_1 + \cdots + a_n$. Prove that there is a bijection $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that none of $a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(j)}$, for $j = 1, \dots, n$, belong to S .

Assume that $a_1 < a_2 < \cdots < a_n$ and we induct on n .

Case 1: Suppose $a_n \in S$. There are $n - 1$ holes $\{a_1, a_1 + a_n\}, \dots, \{a_{n-1}, a_{n-1} + a_n\}$ and at most $n - 2$ pigeons in $S \setminus \{a_n\}$. Hence there is one hole without pigeons. In other words, there exists

some $i = 1, \dots, n-1$ such that $a_i \notin S$ and $a_i + a_n \notin S$. We will let $\sigma(1) = i$ and $\sigma(2) = n$. There are now $n-2$ remaining a_j 's.

Case 1a): Suppose S contains a_k for some $k = 1, \dots, n-1$. Then the set $S' = \{s - a_i - a_n : s > a_i + a_n, s \in S\}$ has at most $n-3$ elements, since s can't be a_k or a_n . By induction, there exists $\sigma' : \{3, \dots, n\} \rightarrow \{1, \dots, i-1, i+1, \dots, n-1\}$ such that none of $a_{\sigma'(3)} + \dots + a_{\sigma'(j)}$, for $j = 3, \dots, n$, belong to S' . Hence, none of $a_i + a_n + a_{\sigma'(3)} + \dots + a_{\sigma'(j)}$, for $j = 3, \dots, n$, belong to S .

Case 1b): Suppose S does not contain any of $a_1, \dots, a_{n-1}$. The set $S' = \{s - a_n : s > a_n, s \in S\}$ has at most $n-2$ elements. By induction, there exists $\sigma' : \{2, \dots, n\} \rightarrow \{1, \dots, n-1\}$ such that none of $a_{\sigma'(2)} + \dots + a_{\sigma'(j)}$, for $j = 2, \dots, n$, belong to S' . Hence, none of $a_n + a_{\sigma'(2)} + \dots + a_{\sigma'(j)}$, for $j = 2, \dots, n$, belong to S , but $a_n \in S$. Since $a_{\sigma'(2)} \notin S$, we may take

$$\sigma(1) = \sigma'(2), \quad \sigma(2) = n, \quad \sigma(j) = \sigma'(j) \text{ for } j \geq 3.$$

Case 2: Suppose $a_n \notin S$. Suppose S contains some element less than a_n . Then the argument of Case 1b) works with none of $a_n + a_{\sigma'(2)} + \dots + a_{\sigma'(j)}$, for $j = 2, \dots, n$, belong to S , and $a_n \notin S$. So we simply take $\sigma(1) = n$ and $\sigma(j) = \sigma'(j)$ for $j \geq 2$.

Case 3: Suppose $a_n \notin S$ and all elements of S are larger than a_n . Denote the elements of S by $s_1 < \dots < s_m$ with $m \leq n-1$. By induction, there exists $\sigma' : \{2, \dots, n\} \rightarrow \{1, \dots, n-1\}$ such that none of $a_{\sigma'(2)} + \dots + a_{\sigma'(j)}$, for $j = 2, \dots, n$, belong to $\{s_2 - a_n, \dots, s_m - a_n\}$. We take $\sigma(1) = n$ and $\sigma(j) = \sigma'(j)$ for $j \geq 2$. Suppose that σ does not work. Then there exists $i = 2, \dots, n$ such that $a_n + a_{\sigma'(2)} + \dots + a_{\sigma'(i)} = s_1$. Note $i \leq n-1$ since S does not contain the full sum. Consider now

$$a_{\sigma'(i+1)}, a_{\sigma'(2)}, \dots, a_{\sigma'(i)}, a_n, a_{\sigma'(i+2)}, \dots, a_{\sigma'(n)}.$$

The partial sums up to and including

$$a_{\sigma'(i+1)} + a_{\sigma'(2)} + \dots + a_{\sigma'(i)} < a_n + a_{\sigma'(2)} + \dots + a_{\sigma'(i)} = s_1$$

so they miss elements of S . The partial sums after and including

$$a_{\sigma'(i+1)} + a_{\sigma'(2)} + \dots + a_{\sigma'(i)} + a_n > a_{\sigma'(2)} + \dots + a_{\sigma'(i)} + a_n = s_1$$

miss all of $s_2, \dots, s_m$. So we are done.