# Week 9

## Mock Putnam problems

A1 Let $p(x) \in \mathbb{C}[x]$ be a polynomial of degree $n > 0$. Let $a \neq b$ be distinct complex numbers. Prove that $(p(x)-a)(p(x)-b)$ has at least $n+1$ distinct roots.

A2 Find the number of non-constant $p(x) = a_n x^n + \cdots + a_0$ with $a_0, \ldots, a_n$ a permutation of $0, 1, \ldots, n$ and that $p(x)$ factors into linear polynomials in $\mathbb{Q}[x]$.

A3 Evaluate the integral

$$\int_0^1 \frac{\ln|x^{69} - (1-x)^{69}|}{x} dx.$$

A4 Define the sequence $(a_n)$ by $a_0 = 1$ and

$$a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{a_k}{n-k+2}.$$

Determine if $\sum_{n=0}^{\infty} a_n/2^n$ converges and if so, find its value.

A5 Let $p$ be a prime and let $d$ be a positive integer such that $\{n^d + n^3 \colon n \in \mathbb{F}_p\} = \mathbb{F}_p$. Find possible values for $2^d \in \mathbb{F}_p$.

A6 Let $G$ be an abelian group with $n$ elements, generated by $k$ elements $g_1, \ldots, g_k$ where $g_1 = e$ is the identity and $k < n$. Let $X_1, X_2, \ldots$ be independent random variables uniform on $\{g_1, \ldots, g_k\}$. Prove that there exists $\lambda \in (0,1)$ such that

$$\lim_{m \to \infty} \frac{1}{\lambda^{2m}} \sum_{g \in G} \left( \Pr(X_1 \cdots X_m = g) - \frac{1}{n} \right)^2$$

exists and is positive.