Week 7

Number Theory

1. (2021A5) Let A be the set of all integers n such that $1 \leq n \leq 2025$ and $\gcd(n, 2025) = 1$. For every positive integer j, let
$$S(j) = \sum_{n \in A} n^j.$$
Let $1 \leq j_1 < j_2 < j_3 < \cdots$ denote all the positive integers j such that $S(j)$ is not divisible by 2025. Find j_{145}.

2. (2023A5) Let $m \in \mathbb{N}$. For any non-negative integer k, let $f(k)$ denote the number of 1's in the base-3 representation of k. Find all $z \in \mathbb{C}$ such that
$$\sum_{k=0}^{3^m-1} (-2)^{f(k)} (z+k)^{2m+3} = 0.$$

Mock Putnam problems

A1 Let $f : \mathbb{Q} \to \mathbb{Z}$ be any function. Does there exist rational numbers $a < b < c$ such that $f(b) = \max\{f(a), f(b), f(c)\}$?

A2 Find
$$\lim_{x \to 0^+} \frac{1}{x \ln x} \sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}.$$

A4 Let X, Y be independent and identically distributed random variables with values in $\mathbb{R}$. Prove that $\Pr(|X+Y| < 1) \leq 3 \Pr(|X-Y| < 1)$.

A5 Let T be the set of positive integers n such that there exists $m_1, \ldots, m_{n-1} \in \mathbb{Z}$ such that
$$m_1 \arctan 1 + \cdots + m_{n-1} \arctan(n-1) = \arctan n.$$
Determine if 16 or 17 belongs to T.

Number Theory

1. (2021A5) Let A be the set of all integers n such that $1 \leq n \leq 2025$ and $\gcd(n, 2025) = 1$. For every positive integer j, let

$$S(j) = \sum_{n \in A} n^j.$$

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For any odd positive integer m, let

$$S_m(j) = \sum_{n \in (\mathbb{Z}/m\mathbb{Z})^\times} n^j \in \mathbb{Z}/m\mathbb{Z}.$$

Suppose first m is an odd prime power. Then there exists primitive $\alpha \in (\mathbb{Z}/m\mathbb{Z})^\times$ (so that $o(\alpha) = \phi(m)$). Since $\alpha^j S_p(j) = S_p(j)$, we see that

$$S_m(j) = \begin{cases} 0 & \text{if } \phi(m) \nmid j, \\ \phi(m) & \text{if } \phi(m) \mid j. \end{cases}$$

Suppose now $m = p^k n$ where p is an odd prime, $k \geq 1$ and $p \nmid n$. Then

$$S_m(j) \bmod p^k = \phi(n) S_{p^k}(j).$$

Consider the special case $m = 2025 = 3^4 5^2$. Then

$$\begin{aligned} S(j) \bmod 3^4 &= 5 \cdot 4 \cdot S_{3^4}(j) \neq 0 \iff \phi(3^4) \mid j, \\ S(j) \bmod 5^2 &= 3^3 \cdot 2 \cdot S_{5^2}(j) \neq 0 \iff \phi(5^2) \mid j. \end{aligned}$$

Hence $2025 \nmid S(j)$ if and only if $54 \mid j$ or $20 \mid j$. The number of such $j = 1, \ldots, N$ is

$$j(N) = \left\lfloor \frac{N}{54} \right\rfloor + \left\lfloor \frac{N}{20} \right\rfloor - \left\lfloor \frac{N}{540} \right\rfloor.$$

Note that if $N = 540k$ for some positive integer k, then

$$j(N) = 10k + 27k - k = 36k.$$

Taking $k = 4$ gives $j_{144} = 2160$. Then it's easy to see that $j_{145} = 2180$.

2. (2023A5) Let $m \in \mathbb{N}$. For any non-negative integer k, let $f(k)$ denote the number of 1's in the base-3 representation of k. Find all $z \in \mathbb{C}$ such that

$$\sum_{k=0}^{3^m-1} (-2)^{f(k)} (z+k)^{2m+3} = 0.$$

Define $\chi : \{0, 1, 2\} \to \mathbb{R}$ by

$$\chi(2) = 1, \qquad \chi(1) = -2, \qquad \chi(0) = 1.$$

Then for $k = \overline{a_{m-1}\cdots a_1 a_0}$ in base 3, we have

$$(-2)^{f(k)} = \chi(a_{m-1})\cdots\chi(a_1)\chi(a_0).$$

Expanding

$$\sum_{a_{m-1}=0}^{2}\cdots\sum_{a_0=0}^{2}\chi(a_{m-1})\cdots\chi(a_0)(z+3^{m-1}a_{m-1}+\cdots+a_0)^{2m+3},$$

we see that the coefficient for z^d is

$$\sum_{n_0+\cdots+n_{m-1}=2m+3-d}\binom{2m+3}{d,n_0,\ldots,n_{m-1}}\prod_{i=0}^{m-1}\sum_{a_i=0}^{2}\chi(a_i)(3^i a_i)^{n_i}.$$

When $n_i = 0$, we simply have $\sum_{a=0}^{2}\chi(a) = 0$. However, for $n_i = k > 0$, we have

$$\sum_{a=0}^{2}\chi(a)a^k = 2^k - 2$$

which is unruly. The trick here is to note that

$$\sum_{a=0}^{2}\chi(a)(a-1)^k = 1 + 1(-1)^k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even.} \end{cases}$$

Let $w = z + 3^{m-1} + \cdots + 1$. Then the coefficients of w^d is

$$\sum_{\substack{n_0+\cdots+n_{m-1}=2m+3-d \\ n_0,\ldots,n_{m-1}>0 \\ n_0,\ldots,n_{m-1}\equiv 0 \pmod 2}}\binom{2m+3}{d,n_0,\ldots,n_{m-1}}\prod_{i=0}^{m-1}3^{in_i}\cdot 2.$$

Hence we see that only $d = 1$ and $d = 3$ are possible.

Suppose $d = 3$ first. Then $n_0 = \cdots = n_{m-1} = 2$. The coefficient is

$$\frac{(2m+3)!}{3!(2!)^m}3^{(m-1)m}\cdot 2^m.$$

Suppose next $d = 1$. Then there is a unique $k = 0,\ldots,m-1$ such that $n_k = 4$ and all the other $n_i = 2$. Hence the coefficient is

$$\frac{(2m+3)!}{(2!)^{m-1}4!}\sum_{k=0}^{m-1}2^m 3^{(m-1)m+2k} = \frac{(2m+3)!}{3!(2!)^m}3^{(m-1)m}\cdot 2^m\cdot\frac{9^m-1}{16}.$$

Combining, we find that the polynomial is

$$\frac{(2m+3)!}{3!(2!)^m}3^{(m-1)m}\cdot 2^m\left(w^3 + \frac{9^m-1}{16}w\right).$$

Hence, the three roots are

$$\frac{3^m-1}{2}, \qquad \frac{3^m-1}{2} \pm i\frac{\sqrt{9^m-1}}{4}.$$

Mock Putnam problems

A1 Let $f : \mathbb{Q} \to \mathbb{Z}$ be any function. Does there exist rational numbers $a < b < c$ such that $f(b) = \max\{f(a), f(b), f(c)\}$?

Suppose not. Suppose first there exists $a < b$ such that $f(a) \leq f(b)$. Then for every $c > b$, we have $f(b) < f(c)$. Let $c_1 > b$ such that $f(c_1)$ is minimal. Then for every $r \in (b, c_1)$, we have $f(b) < f(r) \geq f(c_1)$. So (b, r, c_1) does the job.

Hence, we must have $f(a) > f(b)$ for every $a < b$. We reach a similar contradiction by taking $a < 0$ with $f(a)$ minimal. Then for every $r \in (a, 0)$, we have $f(a) \leq f(r)$ but since $a < r$, we should have $f(a) > f(r)$.

A2 Find

$$\lim_{x \to 0^+} \frac{1}{x \ln x} \sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}.$$

Fix small $\epsilon > 0$. Suppose $nx < \epsilon$. Then there exists $y_{n,x} \in (0, \epsilon)$ such that

$$\sin nx = nx - \frac{1}{2}(nx)^2 \sin y_{n,x} \qquad \text{so} \qquad (1-\epsilon)nx < \sin nx < nx.$$

Note that

$$\sum_{n < \epsilon/x} \frac{nx}{n^2} = x(\ln(\epsilon/x) + O(1)) = -x \ln x + x \ln \epsilon + O(x).$$

On the other hand

$$\sum_{n > \epsilon/x} \frac{|\sin nx|}{n^2} \leq \sum_{n > \epsilon/x} \frac{1}{n^2} = O(x/\epsilon).$$

Therefore, we have

$$\sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2} = -x \ln x + O_\epsilon(\epsilon x \ln x) + O(x/\epsilon).$$

Dividing by $x \ln x$, letting $x \to 0^+$, then letting $\epsilon \to 0^+$ gives that the desired limit is -1.

A4 Let X, Y be independent and identically distributed random variables with values in $\mathbb{R}$. Prove that $\Pr(|X + Y| < 1) \leq 3 \Pr(|X - Y| < 1)$.

For each $k \in \mathbb{Z}$, let $p_k = \Pr(k < X < k+1)$. If $k \leq X < k+1$ and $-1 < X + Y < 1$, then $-2 - k < Y < 1 - k$. Hence

$$\begin{aligned}
\Pr(|X+Y| < 1) &\leq \sum_k \Pr(-2-k < Y < 1-k) p_k \\
&= \sum_k (p_{-2-k} + p_{-1-k} + p_{-k}) p_k \\
&\leq \sum_k \frac{1}{2}\left(p_{-2-k}^2 + p_k^2 + p_{-1-k}^2 + p_k^2 + p_{-k}^2 + p_k^2\right) \\
&= 3 \sum_k p_k^2 \\
&= 3 \sum_k \Pr(k < X < k+1, k < Y < k+1) \\
&\leq 3 \Pr(|X - Y| < 1).
\end{aligned}$$

A5 Let T be the set of positive integers n such that there exists $m_1, \ldots, m_{n-1} \in \mathbb{Z}$ such that

$$m_1 \arctan 1 + \cdots + m_{n-1} \arctan(n-1) = \arctan n.$$

Determine if 16 or 17 belongs to T.

We note that $n \in T$ if and only if there exists $m_1, \ldots, m_{n-1} \in \mathbb{Z}$ and $Q \in \mathbb{Q}$ such that

$$1 + ni = Q(1+i)^{m_1} \cdots (1+(n-1)i)^{m_{n-1}}.$$

Let z denote the right hand side. Note that we don't need to worry about the signs of Q since $(1+i)^4 = -4$.

When $n = 16$, we know that $n^2 + 1 = 257$ is a prime, so $1 + ni$ is a Gaussian prime that does not divide any of $1 + mi$ for $m = 1, \ldots, 15$. We can't have $1 + 16i \mid Q$ for then $1 - 16i \mid Q$ but $1 - 16i \nmid 1 + 16i$. So $16 \notin T$.

When $n = 17$, we have $n^2 + 1 = 290 = 2 \times 5 \times 29$. One finds

$$1 + 17i = (5 - 2i)(1 + 2i)(1 + i).$$

(The actual factorization is not needed for the conclusion.) Since $17 \equiv -12 \pmod{29}$, we will take care of the $5 - 2i$ factor using $1 + 12i$. We have $1 + 12^2 = 145 = 5 \cdot 29$. One finds

$$1 + 12i = (5 + 2i)(1 + 2i).$$

(Note that $5 - 2i \nmid 5$ and $1 + 2i \mid 5$, so $1 + 17i$ and $1 + 12i$ share $1 + 2i$ as a common factor but not $5 + 2i$.) Now

$$\frac{29}{1 + 12i} = (5 - 2i)(1 + 2i)^{-1}.$$

Hence, we have

$$1 + 17i = 29(1 + i)(1 + 2i)^2(1 + 12i)^{-1}$$

which translates to

$$\arctan 1 + 2 \arctan 2 - \arctan 12 = \arctan 17.$$