Week 6

Integrals and derivatives

1. (2017A3) Let $a < b$ be real numbers, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{f(x)^{n+1}}{g(x)^n} dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

2. (2019A6) Let g be a real-valued function that is continuous on the closed interval $[0, 1]$ and twice differentiable on the open interval $(0, 1)$. Suppose that for some real number $r > 1$,

$$\lim_{x\to 0^+} \frac{g(x)}{x^r} = 0.$$

Prove that either

$$\lim_{x\to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x\to 0^+} x^r |g''(x)| = \infty.$$

Mock Putnam problems

A2 Find the largest $C \in \mathbb{R}$ such that for any twice differentiable function $f : [0, 1] \to \mathbb{R}$ with $2\displaystyle\int_0^1 f(x)dx = f(1)$, we have

$$\int_0^1 (f''(x))^2 dx \geq C f(0)^2.$$

A3 Find all $a \in \mathbb{R}$ such that $f(x) = x^4 - ax + 1$ can be written as $P(x)/Q(x)$ where P, Q are polynomial with non-negative real coefficients.

A4 Let $a, b, c, d \in \mathbb{Z}$ with $ad \neq 0$ and $b^2 - 4ac = d^2a^2$. Suppose $an^2 + bn + c \neq 0$ for any $n \in \mathbb{N}$. Determine if

$$\sum_{n=1}^{\infty} \frac{1}{an^2 + bn + c} \in \mathbb{Q}.$$

A6 Let $0 < p < 1$. Random variables $X_1, X_2, \ldots$ with values in the positive integers are chosen so that $\Pr(X_1 = k) = (1 - p)p^{k-1}$; and given $X_1, \ldots, X_n$, order the remaining positive integers as $m_1 < m_2 < \cdots$, then $\Pr(X_{n+1} = m_k) = (1 - p)p^{k-1}$. Let $Y_n = \max\{X_1, \ldots, X_n\}$. For any positive integer m, let $d(m)$ is the number of positive divisors of m. Prove that

$$\lim_{n\to\infty} E(Y_n - n) = \sum_{m=1}^{\infty} d(m)\, p^m.$$

Integrals and derivatives

1. (2017A3) Let $a < b$ be real numbers, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x)dx = \int_a^b g(x)dx$ but $f \neq g$. For every positive integer n, define

$$I_n = \int_a^b \frac{f(x)^{n+1}}{g(x)^n} dx.$$

Show that $I_1, I_2, I_3, \ldots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

The first thing that comes to mind from $f \neq g$ is

$$\int_a^b (f-g)^2 dx = \int_a^b f^2 - 2fg + g^2 dx > 0.$$

This doesn't quite help but if we divide by g, we have

$$\int_a^b \frac{(f-g)^2}{g} dx = \int_a^b \frac{f^2}{g} - 2f + g\, dx = I_1 - I_0 > 0.$$

We now multiply by f^n/g^n to get

$$\int_a^b \frac{f^n}{g^n}\frac{(f-g)^2}{g} dx = \int_a^b \frac{f^{n+2}}{g^{n+1}} - 2\frac{f^{n+1}}{g^n} + \frac{f^n}{g^{n-1}}\, dx = I_{n+1} - 2I_n + I_{n-1} > 0.$$

Hence we have $I_{n+1} - I_n > I_n - I_{n-1} > 0$ and

$$\sum_{n=1}^{\infty} I_n - I_{n-1} = \lim_{n\to\infty} I_n - I_0 = \infty.$$

2. (2019A6) Let g be a real-valued function that is continuous on the closed interval $[0, 1]$ and twice differentiable on the open interval $(0, 1)$. Suppose that for some real number $r > 1$,

$$\lim_{x\to 0^+} \frac{g(x)}{x^r} = 0.$$

Prove that either

$$\lim_{x\to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x\to 0^+} x^r |g''(x)| = \infty.$$

Suppose $\limsup_{x\to 0^+} x^r|g''(x)| < \infty$. This means that $x^r|g''(x)|$ is bounded, say by M, for x close to 0. Suppose for a contradiction that for some $\alpha > 0$, there exists x_0 arbitrarily close to 0 such that $|g'(x_0)| > \alpha$. For Δx small enough (so that $x_0 + \Delta x$ is close enough to 0 for the bound of $x^r|g''(x)| \leq M$), we have for any $x \in (x_0, x_0 + \Delta x)$,

$$|g'(x) - g'(x_0)| \leq \frac{M}{x_0^r}\Delta x.$$

By taking

$$\Delta x = \frac{x_0^r}{M}\frac{\alpha}{2}$$

(shrink x_0 if necessary to make sure this is less than $x_0/1000$, which is possible since $r > 1$), we have $|g'(x)| > \alpha/2$ for every $x \in (x_0, x_0 + \Delta x)$. Fix any $\delta > 0$, by taking x_0 small enough, we have $|g(x)| < \delta x^r$ for every $x \in (0, x_0 + \Delta x)$. For some $x \in (x_0, x_0 + \Delta x)$, we have

$$|g'(x)| = \frac{|g(x_0 + \Delta x) - g(x_0)|}{\Delta x} \leq \frac{(1001^r/1000^r + 1)\delta x_0^r}{\Delta x} = C'\delta/\alpha$$

for some constant $C' > 0$, depending on r and M. Letting $\delta \to 0^+$ gives the contradiction.

Mock Putnam problems

A2 Find the largest $C \in \mathbb{R}$ such that for any twice differentiable function $f : [0,1] \to \mathbb{R}$ with $2\int_0^1 f(x)dx = f(1)$, we have

$$\int_0^1 (f''(x))^2 dx \geq Cf(0)^2.$$

Consider an inner product on continuous functions on $[0,1]$ by $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. Then to find lower bounds for $\|f''\|^2$, the general strategy is to project onto the subspace spanned by polynomials. We compute

$$\begin{aligned} \int_0^1 f''(x)dx &= f'(1) - f'(0), \\ \int_0^1 xf''(x)dx &= f'(1) - f(1) + f(0), \\ \int_0^1 x^2 f''(x)dx &= f'(1) - f(1). \end{aligned}$$

We note that

$$\int_0^1 (x^2 - x)f''(x)dx = -f(0).$$

We can then forget about the above general strategy and apply Cauchy-Schwartz:

$$f(0)^2 = \left|\int_0^1 (x^2 - x)f''(x)dx\right|^2 \leq \int_0^1 (x^2 - x)^2 dx \cdot \int_0^1 f''(x)^2 dx = \frac{1}{30}\int_0^1 f''(x)^2 dx.$$

Equality is achieved when $f''(x) = x^2 - x$. Solving for $f(x)$ gives that $f(x) = x^4/12 - x^3/6 - 1/30$ satisfies the given condition.

A3 Find all $a \in \mathbb{R}$ such that $f(x) = x^4 - ax + 1$ can be written as $P(x)/Q(x)$ where P, Q are polynomial with non-negative real coefficients.

We may consider only $a > 0$. A polynomial with non-negative real coefficients has no positive real roots. Solving $f'(x) = 0$ gives $x_0 = (a/4)^{1/3} > 0$ and $f(x_0) = -(3a/4)x_0 + 1$. If $f(x_0) < 0$, then f has a positive real root. Hence we have $a < 4 \cdot 3^{-3/4}$.

Conversely, suppose $a < 4 \cdot 3^{-3/4}$. Then it suffices to find N so that $(x^4 + 1)^{4N} - (ax)^{4N}$ has non-negative coefficients. We use Stirling's approximation $m! \sim \sqrt{2\pi m}(m/e)^m$ to find

$$\binom{4N}{N}^{1/N} \sim \frac{(4N/e)^4}{(N/e)(3N/e)^3} = \frac{4^4}{3^3} > a^4.$$

Hence
$$\binom{4N}{N} > a^{4N}$$
for N large enough.

Without Stirling's formula, the estimate
$$\ln(n!) = n\ln n - n + O(\ln n)$$
can be obtained from
$$\sum_{k=1}^{n} \ln k = \int_1^n \ln x\,dx + O\left(\sum_{k=1}^{n} \frac{1}{k}\right).$$
Then we have, for any $k \in N$,
$$\frac{1}{n}\ln((kn)!) = k\ln n + k\ln k - k + O\Big(\frac{\ln k + \ln n}{n}\Big).$$
Therefore,
$$\lim_{n\to\infty} \frac{1}{n}\ln\left(\binom{kn}{rn}\right) = k\ln k - r\ln r - (k-r)\ln(k-r).$$

A4 Let $a, b, c, d \in \mathbb{Z}$ with $ad \neq 0$ and $b^2 - 4ac = d^2a^2$. Suppose $an^2 + bn + c \neq 0$ for any $n \in \mathbb{N}$. Determine if
$$\sum_{n=1}^{\infty} \frac{1}{an^2 + bn + c} \in \mathbb{Q}.$$

We may assume $d > 0$. The two roots of $ax^2 + bx + c$ are
$$r_1 = \frac{-b - da}{2a} \qquad \text{and} \qquad r_2 = \frac{-b + da}{2a}.$$
In other words, $r_1 - r_2 = d \in \mathbb{Z}$. Then
$$\frac{1}{an^2 + bn + c} = \frac{1}{a(r_2 - r_1)}\left(\frac{1}{n - r_1} - \frac{1}{n - r_2}\right).$$
Let N be large enough so that $N - r_1 - 1 \geq 0$. Hence
$$\begin{aligned}
\sum_{n=N}^{\infty} \frac{1}{an^2 + bn + c} &= \frac{1}{ad}\sum_{n=N}^{\infty}\int_0^1 x^{n-r_2-1} - x^{n-r_1-1}dx \\
&= \frac{1}{ad}\int_0^1 (x^{N-r_2-1} - x^{N-r_1-1})\sum_{n=0}^{\infty} x^n dx \\
&= \frac{1}{ad}\int_0^1 x^{N-r_1-1}\frac{x^d - 1}{1 - x}dx \\
&\in \mathbb{Q}.
\end{aligned}$$

A6 Let $0 < p < 1$. Random variables $X_1, X_2, \dots$ with values in the positive integers are chosen so that $\Pr(X_1 = k) = (1-p)p^{k-1}$; and given $X_1, \dots, X_n$, order the remaining positive integers as $m_1 < m_2 < \cdots$, then $\Pr(X_{n+1} = m_k) = (1-p)p^{k-1}$. Let $Y_n = \max\{X_1, \dots, X_n\}$. For any positive integer m, let $d(m)$ is the number of positive divisors of m. Prove that

$$\lim_{n\to\infty} E(Y_n - n) = \sum_{m=1}^{\infty} d(m)\, p^m.$$

Note first that

$$E(Y_1) = E(X_1) = \sum_{k>0} k(1-p)p^{k-1} = \frac{1}{1-p}.$$

The first attempt looks at

$$E(Y_{n+1} - Y_n \mid Y_n) = \sum_{k>Y_n} (1-p)p^{k-1-n}(k - Y_n) = \sum_{j>0} (1-p)jp^{j-1}p^{Y_n - n} = \frac{p^{Y_n-n}}{1-p}$$

which is a bit unruly. However, note that $Y_{n+1} - Y_n | X_{n+1} > Y_n$ has the same distribution as X_1. Hence

$$E(Y_{n+1} - Y_n \mid X_{n+1} > Y_n) = E(X_1) = \frac{1}{1-p}.$$

To find $\Pr(X_{n+1} > Y_n)$, we note that given distinct $a_1, \dots, a_n$ and $b > \max\{a_1, \dots, a_n\}$, we have

$$\begin{aligned} \Pr((X_1, \dots, X_{n+1}) = (b, a_1, a_2, \dots, a_n)) &= p \cdot \Pr((X_1, \dots, X_{n+1}) = (a_1, b, a_2, \dots, a_n)) \\ &= p^2 \cdot \Pr((X_1, \dots, X_{n+1}) = (a_1, a_2, b, \dots, a_n)) \\ &= p^n \cdot \Pr((X_1, \dots, X_{n+1}) = (a_1, a_2, \dots, a_n, b)) \end{aligned}$$

Hence, we see that

$$\Pr(X_{n+1} > Y_n) = \frac{1}{1 + p + p^2 + \cdots + p^n} = \frac{1-p}{1-p^{n+1}}.$$

Therefore, we have

$$E(Y_{n+1} - Y_n) = \frac{1}{1-p^{n+1}} \qquad \text{so} \qquad E(Y_n) = \sum_{k=1}^{n} \frac{1}{1-p^k}.$$

Then we have

$$E(Y_n - n) = \sum_{k=1}^{n} \frac{p^k}{1-p^k} \to \sum_{k=1}^{\infty} \sum_{d=1}^{\infty} p^{kd} = \sum_{m=1}^{\infty} d(m)\, p^m.$$