Week 5

Sequences

1. (2020A3) Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether $\sum_{n=1}^{\infty} a_n^2$ converges.

2. (2012B4) Let $a_0 = 0$ and let $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 0$. Prove that $\lim_{n\to\infty} a_n - \log n = 0$.

Mock Putnam problems

A2 Find all $m \in \mathbb{R}$ such that

$$\lim_{x\to 0^+} \frac{1}{x^m} \int_0^x \left| \sin \frac{1}{t} \right| dt$$

exists and also compute the value.

A3 Let $T : \mathbb{Z} \to \mathbb{Z}$ be any map. Prove that there does not exist a non-constant polynomial $P(x)$ with integer coefficients such that for every positive integer n, the number of $x \in \mathbb{Z}$ such that $T^n(x) = x$ is $P(n)$. Here T^n denotes T iterated n times.

A4 Let $v_1, \ldots, v_n \in \mathbb{R}^n$ with length $|v_i| \leq 1$ for all $i = 1, \ldots, n$. Let $p_1, \ldots, p_n \in [0, 1]$ and let $w = p_1 v_1 + \cdots + p_n v_n$. Show that there exist $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ such that

$$|w - (\epsilon_1 v_1 + \cdots + \epsilon_n v_n)| \leq \frac{1}{2}\sqrt{n}.$$

A6 (a) Let $p(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that for any $x_0 \in \mathbb{R}$, $p(x_0, y) \in \mathbb{R}[y]$; and for any $y_0 \in \mathbb{R}$, $p(x, y_0) \in \mathbb{R}[x]$. Prove that $p(x, y) \in \mathbb{R}[x, y]$. Give a counterexample over $\mathbb{Q}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any $y_0 \in \mathbb{R}$, $f(x+y_0) - f(x) \in \mathbb{R}[x]$. Prove that $f(x) \in \mathbb{R}[x]$.

Sequences

1. (2020A3) Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether $\sum_{n=1}^{\infty} a_n^2$ converges.

Since $\sin x \sim x$, we expect that the a_n don't change much and so the series should diverge. This suggests looking for a lower bound. We have

$$\sin x \geq x - x^3/6 + (\sin c)x^4/24 \geq x - x^3/6$$

for some $c \in (0, x)$. So $a_n \geq a_{n-1} - a_{n-1}^3/6$. Note also that $x - x^3/6$ is increasing for $x \in (0, 1)$. We prove $a_n \geq 1/\sqrt{n}$ by induction on n. It suffices to prove

$$\frac{1}{\sqrt{n-1}} - \frac{1}{6(n-1)\sqrt{n-1}} \geq \frac{1}{\sqrt{n}}.$$

Bash. Note that

$$\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} = -\frac{1}{2c\sqrt{c}} \leq -\frac{1}{2n\sqrt{n}}$$

for some $c \in (n-1, n)$. It is easy to check that $3(n-1)^{3/2} \geq n^{3/2}$ as $3^{2/3} \geq 2 \geq \frac{n}{n-1}$.

Extra: The series $\sum_{n=1}^{\infty} a_n^3$ converges. From

$$\sin x \leq x - x^3/6 + x^5/120 \leq x - x^3/6 + x^3/120 = x - cx^3/6$$

for some $c = 19/120 > 0$, we get $a_n \leq a_{n-1} - ca_{n-1}^3$. Hence

$$c\sum_{j=1}^{n-1} a_j^3 \leq a_0 - a_n \leq a_0.$$

Let $\alpha > 0$ be a constant to be chosen later. Let $n_0 \in \mathbb{N}$ be such that $\alpha/\sqrt{n_0} \leq 1$. Let $k \geq 0$ such that $a_{k+n_0} \leq \alpha/\sqrt{n_0}$ and we prove by induction that $a_{n+k} \leq \alpha/\sqrt{n}$ for all $n \geq n_0$. It suffices to prove (since $x - cx^3$ is increasing for $c < 1/6$ and $x \in (0, 1)$)

$$\frac{\alpha}{\sqrt{n-1}} - \frac{c\alpha^3}{(n-1)^{3/2}} \leq \frac{\alpha}{\sqrt{n}}.$$

We have

$$\frac{\alpha}{\sqrt{n}} - \frac{\alpha}{\sqrt{n-1}} \geq -\frac{\alpha}{2n\sqrt{n}}.$$

Hence, it suffices to have

$$\frac{1}{2c\alpha^2} \leq \left(\frac{n}{n-1}\right)^{3/2} \leq 2^{3/2}.$$

So $\alpha = 1.06$, $n_0 = 2$ and $k = 1$ will do, as $\sin(\sin 1) - 1.06/\sqrt{2} = -0.0039...$ In other words, $a_{n+1} \leq 1.06/\sqrt{n}$ for all $n \geq 1$. So the series $\sum_{n=1}^{\infty} a_n^{\ell}$ converges for all $\ell > 2$.

The reason to compare with $1/\sqrt{n}$ is that

$$a_{n+1} - a_n = \sin(a_n) - a_n \approx -a_n^3/6.$$

If we turn this into the differential equation $y' = -y^3/6$, we find that the solution is given by $1/y^2 = x/6 + C$. Hence $y \approx 1/\sqrt{x/6 + C}$.

2. (2012B4) Let $a_0 = 0$ and let $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 0$. Prove that $\lim_{n\to\infty} a_n - \log n = 0$.

Let $b_n = e^{a_n} > 1$ so

$$b_{n+1} = b_n e^{1/b_n} = b_n + 1 + c_n$$

where $0 \leq c_n < C b_n^{-1}$ for some constant $C > 0$. Then we have

$$b_n = n + b_0 + \sum_{i=0}^{n-1} c_i \geq n + b_0.$$

Using this bound for c_i then gives

$$b_n \leq n + b_0 + C \sum_{i=0}^{n-1} \frac{1}{i + b_0} = n + O(\log n).$$

Hence, we find that $b_n/n \to 1$ which is equivalent to $a_n - \log n \to 0$.

Mock Putnam problems

A2 Find all $m \in \mathbb{R}$ such that

$$\lim_{x\to 0^+} \frac{1}{x^m} \int_0^x \left|\sin \frac{1}{t}\right| dt$$

exists and also compute the value.

Let $u = 1/t$ and $y = 1/x$ so we have

$$\lim_{x\to 0^+} \frac{1}{x^m} \int_0^x \left|\sin \frac{1}{t}\right| dt = \lim_{y\to\infty} y^m \int_y^\infty \left|\frac{\sin u}{u^2}\right| du.$$

Let n_y be the smallest integer such that $n_y \pi > y$. Then we have

$$\begin{aligned}
\int_y^\infty \left|\frac{\sin u}{u^2}\right| du &= \int_{n_y\pi}^\infty \left|\frac{\sin u}{u^2}\right| du + \int_y^{n_y\pi} \left|\frac{\sin u}{u^2}\right| du \\
&= 2 \sum_{k\geq n_y} \frac{1}{(k\pi)^2} + O((n_y\pi)^{-3}) + O(y^{-2}) \\
&= \frac{2/\pi^2}{n_y} + O(y^{-2}) \\
&= \frac{2}{\pi} y^{-1} + O(y^{-2})
\end{aligned}$$

since $n_y = y/\pi + O(1)$. We see that the limit doesn't exist for $m > 1$, and goes to 0 for $m < 1$. When $m = 1$, the limit is $2/\pi$.

A3 Let $T : \mathbb{Z} \to \mathbb{Z}$ be any map. Prove that there does not exist a non-constant polynomial $P(x)$ with integer coefficients such that for every positive integer n, the number of $x \in \mathbb{Z}$ such that $T^n(x) = x$ is $P(n)$. Here T^n denotes T iterated n times.

Let a_n denote the number of $x \in \mathbb{Z}$ such that $T^n(x) = x$. Let b_n denote the number of $x \in \mathbb{Z}$ such that n is the smallest positive integer such that $T^n(x) = x$. Then standard division algorithm argument shows that

$$a_n = \sum_{d|n} b_d.$$

Moreover, if $x \in \mathbb{Z}$ such that n is the smallest positive integer such that $T^n(x) = x$, then the same is true for $T(x), T^2(x), \ldots, T^{n-1}(x)$. Hence we have $n \mid b_n$.

Let p, q be distinct primes. Then

$$P(0) \equiv P(pq) = b_1 + b_p + b_q + b_{pq} \equiv b_1 + b_p \pmod{q}.$$

Hence $P(0) = b_1 + b_p = P(p)$ since their difference is divisible by infinitely many primes. Therefore P is constant since $P(x) - P(0)$ has infinitely many roots.

A4 Let $v_1, \ldots, v_n \in \mathbb{R}^n$ with length $|v_i| \leq 1$ for all $i = 1, \ldots, n$. Let $p_1, \ldots, p_n \in [0, 1]$ and let $w = p_1 v_1 + \cdots + p_n v_n$. Show that there exist $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$ such that

$$|w - (\epsilon_1 v_1 + \cdots + \epsilon_n v_n)| \leq \frac{1}{2}\sqrt{n}.$$

Let $X_i = \{0, 1\}$ be independent random variables so that $\Pr(X_i = 1) = p_i$ and $\Pr(X_i = 0) = 1 - p_i$. Let

$$X = |w - (X_1 v_1 + \cdots + X_n v_n)|^2 = \sum_{i,j} (p_i - X_i)(p_j - X_j) v_i^t v_j.$$

Note that

$$E(p_i - X_i) = p_i - E(X_i) = 0$$

and

$$E((p_i - X_i)^2) = p_i(p_i - 1)^2 + (1 - p_i)p_i^2 = p_i(1 - p_i).$$

Hence, we see that

$$E(X) = \sum_{i=1}^{n} p_i(1 - p_i)|v_i|^2 \leq \sum_{i=1}^{n} p_i(1 - p_i) \leq \frac{n}{4}.$$

Therefore, for some values of X_i we have $\sqrt{X} \leq \sqrt{n}/2$.

A6 (a) Let $p(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that for any $x_0 \in \mathbb{R}$, $p(x_0, y) \in \mathbb{R}[y]$; and for any $y_0 \in \mathbb{R}$, $p(x, y_0) \in \mathbb{R}[x]$. Prove that $p(x, y) \in \mathbb{R}[x, y]$. Give a counterexample over $\mathbb{Q}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any $y_0 \in \mathbb{R}$, $f(x+y_0) - f(x) \in \mathbb{R}[x]$. Prove that $f(x) \in \mathbb{R}[x]$.

(a) For any $n \in \mathbb{N}$, let $S_n = \{z \in \mathbb{R} \colon \deg_x p(x, z) \leq n, \deg_y p(z, y) \leq n\}$. Then $\bigcup_{n \in \mathbb{N}} S_n = \mathbb{R}$. Since $\mathbb{R}$ is uncountable, there exists n such that S_n is infinite. (In fact, one can show that each S_n is closed and then use Baire category to say that some S_n contains an open interval.) Fix $z_1, \ldots, z_{n+1} \in S_n$. Using Lagrange interpolation, there is a polynomial $q(x, y)$ of bi-degree (n, n) such that $q(z_i, z_j) = p(z_i, z_j)$ for all $1 \leq i, j \leq n + 1$:

$$q(x, y) = \sum_{i,j} p(z_i, z_j) \frac{\prod_{k \neq i}(x - z_k) \prod_{\ell \neq j}(y - z_\ell)}{\prod_{k \neq i}(z_i - z_k) \prod_{\ell \neq j}(z_j - z_\ell)}.$$

Then for any z_i, we see that $p(z_i, y) = q(z_i, y) \in \mathbb{R}[y]$ since they have degrees at most n and agree on $z_1, \ldots, z_{n+1}$. Now for any $y_0 \in S_n$, we have $p(x, y_0) = q(x, y_0) \in \mathbb{R}[x]$ since they have degrees at most n and agree on $z_1, \ldots, z_{n+1}$. Finally, for any $x_0 \in \mathbb{R}$, we have $p(x_0, y) = q(x_0, y)$ since they have finite degrees and agree on the infinite set S_n.

For a counterexample, take an enumeration $\{r_n\}$ of $\mathbb{Q}$. For any $n \in \mathbb{N}$, let $f_n(x) = (x - r_1)\cdots(x - r_n)$. Take

$$p(x,y) = \sum_{n=1}^{\infty} f_n(x) f_n(y).$$

For any $r = r_m \in \mathbb{Q}$, we have $f_n(r_m) = 0$ for $n \geq m$. Hence

$$p(r_m, y) = \sum_{n=1}^{m-1} f_n(r_m) f_n(y) \in \mathbb{Q}[y]$$

and similarly for $p(x, r_m)$. However, $\deg_y p(r_m, y) = m - 1$ is unbounded, so $p(x,y) \notin \mathbb{Q}[x,y]$.

(b) Let $p(x,y) = f(x+y) - f(x) - f(y) + f(0)$. Then by (a), we see that $p(x,y) \in \mathbb{R}[x,y]$. Let $F(x) = \int_0^x f(t)dt$. Then one checks that

$$F(x+y) - F(x) - F(y) = \int_0^x p(y,s)ds + f(y)x - f(0)x$$

is a polynomial in x for fixed y; and similarly also in the other variable. Then by (a), we have $F(x+y) - F(x) - F(y) = q(x,y)$ and we can solve for $f(y) \in \mathbb{R}[y]$.

Alternative solution by Jason F: For any $g \in \mathbb{R}[x]$ and any $a \in \mathbb{R}$, there is a unique $q(x) \in x\mathbb{R}[x]$ such that $q(x+a) - q(x) = g(x)$. This follows because $\{(x+a)^n - x^n : n \in \mathbb{N}\}$ forms an $\mathbb{R}$-basis for $\mathbb{R}[x]$. For any $n \in \mathbb{N}$, let $q_n(x) \in x\mathbb{R}[x]$ with

$$q_n(x + 1/n) - q_n(x) = f(x + 1/n) - f(x).$$

Then

$$q_n(x+1) - q_n(x) = f(x+1) - f(x) = q_1(x+1) - q_1(x)$$

implying that $q_n = q_1$. We then also have

$$q_1(x + m/n) - q_1(x) = q_n(x + m/n) - q_n(x) = f(x + m/n) - f(x).$$

Hence, we see that $f(r) - q_1(r) = f(0) - q_1(0)$ for any $r \in \mathbb{Q}$. Continuity gives $f(x) = q_1(x) + f(0) - q_1(0) \in \mathbb{R}[x]$.