Week 4

Polynomials

1. (2022A2) Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$.

2. (2021A6) Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as a product of two nonconstant polynomials with integer coefficients. Prove that $P(2)$ is a composite integer.

Mock Putnam problems

A2 A subset $S \subset \mathbb{C}$ is *inverse-free* if there does not exist $x, y \in S$ such that $x + y = 0$. Suppose $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ are two inverse-free subsets of $\mathbb{C}$ such that

$$x_1^{2k-1} + \cdots + x_m^{2k-1} = y_1^{2k-1} + \cdots + y_n^{2k-1} \qquad \text{for all } k = 1, \dots, \max\{m, n\}.$$

Prove that the two sets are equal.

A3 Let $c > 0$ be a real number. Find the supremum of all c such that for any twice differentiable function $f : [0,1] \to \mathbb{R}$ such that

$$f(0) = 0, \qquad f(1) = c, \qquad \int_0^1 f(x)dx = 1,$$

there exists $\eta \in (0,1)$ such that $f''(\eta) < -2$.

A5 For $0 \leq p < 1/2$, let X_i be the random variable with

$$\Pr(X_i = -1) = \Pr(X_i = 1) = p \qquad \text{and} \qquad \Pr(X_i = 0) = 1 - 2p.$$

Prove that if $0 \leq p \leq 1/4$, then for any $n \in \mathbb{N}$, any integers $a_1, \dots, a_n, b$, we have

$$\Pr(a_1X_1 + \cdots + a_nX_n = 0) \geq \Pr(a_1X_1 + \cdots + a_nX_n = b).$$

A6 Let $N(X)$ be the number of pairs (a, b) of positive integers such that $1 < a \leq b \leq X$ and $(a^2 - 1)(b^2 - 1)$ is a perfect square. Find $\lim_{X \to \infty} N(X)/X$.

Polynomials

1. (2022A2) Let n be an integer with $n \geq 2$. Over all real polynomials $p(x)$ of degree n, what is the largest possible number of negative coefficients of $p(x)^2$.

   We observe first that the leading coefficient and the constant coefficient of $p(x)^2$ are non-negative. We consider $p(x) = x^n + g(x) + a$ where $g(x) \in \mathbb{R}[x]$ has degree at most $n-1$ and $a \in \mathbb{R}$. Then
   $$p(x)^2 = x^{2n} + 2ax^n + a^2 + (2x^n + g(x) + 2a)g(x).$$
   To make the most number of negative coefficients through this construction, we take $a = 1$ (note that by scaling, only the sign of a is important) and we take $g(x)$ to have all negative but very small coefficients so that every coefficient of $(2x^n+2)g(x)$ is negative but larger in absolute value than the coefficients of $g(x)^2$. For example, we can take $p(x) = x^n + 1 - \epsilon(x^{n-1} + \cdots + 1)$ and let $\epsilon \to 0^+$. We get $2n-2$ negative coefficients this way.

   Can we have $2n-1$ negative coefficients? This means that
   $$p(x)^2 = x^{2n} + c_{2n-1}x^{2n-1} + \cdots + c_1 x + a^2$$
   where all of $c_i < 0$. Coefficient bash to prove impossibility. Note if we take $q(x) = x^n - 10$ so that $q(x)^2 = x^{2n} - 20x^n + 100$ and add it to our $p(x)^2$ from above, we get a polynomial of the form $p(x)^2 + q(x)^2$ with $2n-1$ negative coefficients. This means when we prove impossibility, plugging in values of x is not enough.

2. (2021A6) Let $P(x)$ be a polynomial whose coefficients are all either 0 or 1. Suppose that $P(x)$ can be written as a product of two nonconstant polynomials with integer coefficients. Prove that $P(2)$ is a composite integer.

   This is a special case of the Cohn's irreducibility criterion: If a prime $p = a_n b^n + \cdots + a_0$ is written in base-b, then $a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ is irreducible.

   Lemma: Suppose $Q(x) \in \mathbb{R}[x]$ has no real root in $[1, \infty)$ and that all of its roots have real part less than $3/2$ and that Q has positive leading coefficient. Then $Q(2) > Q(1) > 0$.

   Proof: It suffices to prove this when $Q(x)$ is irreducible in $\mathbb{R}[x]$. Clear if $Q(x) = x - c \in \mathbb{R}[x]$ with $c < 1$. Suppose now $Q(x) = (x - z)(x - \bar{z})$ with $\mathrm{Re}(z) < 3/2$. Then
   $$Q(2) - Q(1) = 3 - 2\mathrm{Re}(z) > 0.$$
   Corollary: If $Q(x) \in \mathbb{Z}[x]$ and the rest of the above, then $Q(2) \geq 2$.

   It remains to prove that $P(x)$ satisfies the hypothesis of the lemma. Write $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. WLOG we may assume $a_0 = 1$. Since all $a_i \geq 0$, we see that $P(x)$ has no positive real roots. Let $z \in \mathbb{C}$ be a complex root. Then
   $$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z = -1.$$
   Suppose $\mathrm{Re}(z) \geq 3/2$. This means $|z| > 3/2$ is quite large. Note that
   $$z + a_{n-1} = (\mathrm{Re}(z) + a_{n-1}) + i\mathrm{Im}(z)$$
   which implies that $|z + a_{n-1}| \geq |z|$. We multiply by z^{-1} to get
   $$1 \leq |1 + a_{n-1}z^{-1}| \leq |z|^{-2} + \cdots + |z|^{-n} \leq \frac{|z|^{-2}}{1 - |z|^{-1}}.$$

This implies $|z|^2 - |z| \le 1$ and so $|z| \le \frac{1+\sqrt{5}}{2} \simeq 1.618$. However, this means z is very close to the positive x-axis. In particular, $\mathrm{Re}(z^{-1})$ and $\mathrm{Re}(z^{-2})$ are both positive. (In fact, $\mathrm{Re}(z^{-3})$ and $\mathrm{Re}(z^{-4})$ are also positive. So

$$1 \le |1 + a_{n-1}z^{-1} + a_{n-2}z^{-2}| \le |z|^{-3} + \cdots + |z|^{-n} \le \frac{|z|^{-3}}{1-|z|^{-1}}.$$

So $|z|^2 - |z| \le |z|^{-1} < 2/3$ but $(|z| - 1/2)^2 - 1/4 > 3/4$.

Mock Putnam problems

A2 A subset $S \subset \mathbb{C}$ is *inverse-free* if there does not exist $x, y \in S$ such that $x + y = 0$. Suppose $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ are two inverse-free subsets of $\mathbb{C}$ such that

$$x_1^{2k-1} + \cdots + x_m^{2k-1} = y_1^{2k-1} + \cdots + y_n^{2k-1} \qquad \text{for all } k = 1, \ldots, \max\{m, n\}.$$

Prove that the two sets are equal.

(Adapted from 2022A6) Suppose $m \le n$. Consider

$$p(x) = (x - x_1)\cdots(x - x_m)(x + y_1)\cdots(x + y_n).$$

Let s_k be the sum of the k-th power of the roots of $p(x)$ (counted with multiplicities) and let σ_k denote the k-th elementary symmetric polynomial evaluated at the roots of $p(x)$ (counted with multiplicities). Then we have

$$s_1 = s_3 = \cdots = s_{2n-1} = 0.$$

From the Newton relations, we also find that

$$\sigma_1 = \sigma_3 = \cdots = \sigma_{2n-1} = 0.$$

If $m + n$ is odd, then $m + n \le 2n - 1$ and $\sigma_{m+n} = 0$, contradicting the assumption that none of the x_i, y_i are 0. Hence $m + n$ is even and we find that $p(x)$ is an even polynomial.

A3 Let $c > 0$ be a real number. Find the supremum of all c such that for any twice differentiable function $f : [0, 1] \to \mathbb{R}$ such that

$$f(0) = 0, \qquad f(1) = c, \qquad \int_0^1 f(x)dx = 1,$$

there exists $\eta \in (0, 1)$ such that $f''(\eta) < -2$.

Let $h_c(x) = (3c - 6)x^2 - (2c - 6)x$. Then $h_c(x)$ satisfies all the desired conditions and $h_c''(x) = 6(c - 2)$. Hence $h_c(x)$ is a counterexample if $c \ge 5/3$.

Suppose now $c < 5/3$. Then $h_c''(x) < -2$ for all $x \in (0, 1)$. Let $g(x) = f(x) - h_c(x)$. Then

$$g(0) = g(1) = \int_0^1 g(x)dx = 0.$$

It is now easy to check that there exists $\eta \in (0, 1)$ such that $g''(\eta) \le 0$ (and also some $\xi \in (0, 1)$ where $g''(\xi) \le 0$). Indeed, we may assume that $g(x)$ is not identically 0. Then there exists $a, b \in (0, 1)$ such that $g(a) < 0$ and $g(b) > 0$. Suppose first $a < b$. Then g' hits some negative value in $(0, a)$, some positive value in (a, b), and some negative value in $(b, 1)$. So g'' takes both positive and negative values. Similarly when $a > b$.

A5 For $0 \le p < 1/2$, let X_i be the random variable with

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Prove that if $0 \le p \le 1/4$, then for any $n \in \mathbb{N}$, any integers $a_1, \dots, a_n, b$, we have

$$\Pr(a_1 X_1 + \cdots + a_n X_n = 0) \ge \Pr(a_1 X_1 + \cdots + a_n X_n = b).$$

For a random variable X with finite support, we define it's "Fourier transform" by

$$F(\alpha; X) = \sum_k \Pr(X = k) e(k\alpha) = E(e(X\alpha))$$

where $e(z) = e^{2\pi i z}$. Then we have

$$F(\alpha; X_i) = pe(-\alpha) + pe(\alpha) + (1 - 2p) = 1 - 2p + 2p\cos(2\pi\alpha) \ge 0$$

for all $\alpha \in \mathbb{R}$. Hence, we have

$$F(\alpha; a_1 X_1 + \cdots a_n X_n) = \prod_{i=1}^n F(a_i \alpha; X_i) \ge 0.$$

We are now done because

$$\Pr(X = b) = \int_0^1 F(\alpha; X) e(-b\alpha) d\alpha \le \int_0^1 |F(\alpha; X)| d\alpha = \Pr(X = 0)$$

if $F(\alpha; X) \ge 0$ for all α.

A6 Let $N(X)$ be the number of pairs (a, b) of positive integers such that $1 < a \le b \le X$ and $(a^2 - 1)(b^2 - 1)$ is a perfect square. Find $\lim_{X \to \infty} N(X)/X$.

Note that $(a^2-1)(b^2-1)$ is perfect square if and only if $a^2 - 1 = dn^2$ and $b^2 - 1 = dm^2$ for some squarefree positive integer d and $n, m \in \mathbb{N}$. Solutions to the Pell equations $x^2 - dy^2 = 1$ are given by $x + y\sqrt{d} = \pm u^{\mathbb{Z}}$ for a fundamental unit $u \in \mathbb{Z}[\sqrt{d}]^\times$. We may assume $u = x_0 + y_0\sqrt{d}$ where $x_0, y_0 \in \mathbb{N}$. For any $k \in \mathbb{N}$, we write $x_k + y_k\sqrt{d} = u^k$. So

$$x_k \approx y_k \sqrt{d} \approx |u|^k/2 \gg_k d^{k/2}.$$

This is some kind of a "gap principle": two consecutive solutions differ by a factor of at least $\sqrt{d}$. We will use this to show that the main contribution only comes from $a = b$.

Let $N_d(X)$ denote the set of positive integers a such that $1 < a \le X$ with $a^2 - 1 = dn^2$ for some positive integer n. Firstly, if $d \gg X$, then $\#N_d(X) \le 1$. Moreover, we have the "trivial bound" $\#N_d(X) \ll \log X / \log d$. Let $\delta > 0$. For $d < X^{1-\delta}$, we use the trivial bound to obtain

$$\#\{(a, b) \colon a, b \in N_d(X), 2 \le d \le X^{1-\delta}\} \ll X^{1-\delta}(\log X)^2.$$

It remains to consider $X^{1-\delta} < d \ll X$. Then the integer n satisfies $n \ll X^{(1+\delta)/2}$. For every prime power p^k, the number of solutions to $x^2 = 1$ in $\mathbb{Z}/p^k\mathbb{Z}$ is $O(1)$. Hence the number of choices of a mod n^2 is n^ϵ. Once a, n are fixed, d is also fixed and then we'd have X^ϵ choices for b. Hence the contribution for $X^\delta \ll n \ll X^{(1+\delta)/2}$ is

$$\ll \sum_{X^\delta \ll n \ll X^{(1+\delta)/2}} \left(\left\lfloor \frac{X}{n^2} \right\rfloor + 1 \right) n^\epsilon \ll X^{1-\delta+\epsilon} + X^{(1+\delta)/2+\epsilon}$$

which is negligible. Finally, for $n \ll X^\delta$ and $d \ll X$, we have $a \ll X^{(1+\delta)/2}$. For any fixed a, there are X^ϵ choices for d (and n) and then X^ϵ choices for b. So the contribution is once again negligible.