Week 3

Probability

1. (2022A4) Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.

2. (2024B4) Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from $\{1, \ldots, n\}$ and let

$$a_{n,k+1} = \begin{cases} a_{n,k} + 1 & \text{if } m_{n,k} > a_{n,k}, \\ a_{n,k} & \text{if } m_{n,k} = a_{n,k}, \\ a_{n,k} - 1 & \text{if } m_{n,k} < a_{n,k}. \end{cases}$$

Let $E(n)$ be the expected value of $a_{n,n}$. Find $\lim_{n\to\infty} E(n)/n$.

Mock Putnam problems

A1 Let p be an odd prime. Determine the number of functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying:

- $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$;
- if $m \equiv n \pmod{p}$, then $f(m) = f(n)$.

A2 Let p be an odd prime. Let $g(x) = \displaystyle\sum_{k=1}^{p-1} k^{(p-1)/2} x^k \in \mathbb{F}_p[x]$. Find the largest integer n such that $(x-1)^n \mid g(x)$ in $\mathbb{F}_p[x]$.

A4 Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n)$ for $n \geq 1$. Find

$$\lim_{n\to\infty} \frac{n(1 - nx_n)}{\ln n}.$$

A6 Let $X_1, X_2, \ldots$ be uniformly and independently chosen from $[0, 1]$. Let p_n be the probability that $X_i + X_{i+1} \leq 1$ for all $i \leq n - 1$. Find $\lim_{n\to\infty} p_n^{1/n}$.

Probability

1. (2022A4) Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S.

   Define
   $$Y_i = \begin{cases} X_i & \text{if } X_1 \geq X_2 \geq \cdots \geq X_i \\ 0 & \text{otherwise} \end{cases}.$$
   Then $S = \sum_{i=1}^{\infty} Y_i/2^i$. By linearity, it remains to compute $E(Y_i)$. We have
   $$\begin{aligned} E(Y_i) &= \int_0^1 x_i \frac{(1-x_i)^{i-1}}{(i-1)!} dx_i \\ &= \frac{1}{(i-1)!} \int_0^1 (1-x_i)^{i-1} - (1-x_i)^i dx_i \\ &= \frac{1}{(i-1)!} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \frac{1}{(i+1)!} \end{aligned}$$
   Hence
   $$E(S) = \sum_{i=1}^{\infty} \frac{2}{2^{i+1}} \frac{1}{(i+1)!} = 2(e^{1/2} - 1 - 1/2) = 2e^{1/2} - 3.$$

2. (2024B4) Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from $\{1, \ldots, n\}$ and let
   $$a_{n,k+1} = \begin{cases} a_{n,k} + 1 & \text{if } m_{n,k} > a_{n,k}, \\ a_{n,k} & \text{if } m_{n,k} = a_{n,k}, \\ a_{n,k} - 1 & \text{if } m_{n,k} < a_{n,k}. \end{cases}$$
   Let $E(n)$ be the expected value of $a_{n,n}$. Find $\lim_{n\to\infty} E(n)/n$.

   We have
   $$E(a_{n,k+1}|a_{n,k}) = a_{n,k} + \frac{n - a_{n,k}}{n} - \frac{a_{n,k} - 1}{n} = \left(1 - \frac{2}{n}\right) a_{n,k} + \left(1 + \frac{1}{n}\right).$$
   Taking expectation gives
   $$E(a_{n,k+1}) = \left(1 - \frac{2}{n}\right) E(a_{n,k}) + \left(1 + \frac{1}{n}\right).$$
   Using $E(a_{n,0}) = 1$, we get
   $$\begin{aligned} E(a_{n,n}) &= \left(1 - \frac{2}{n}\right)^n + \left(1 + \frac{1}{n}\right) \sum_{k=0}^{n-1} \left(1 - \frac{2}{n}\right)^k \\ &= \left(1 - \frac{2}{n}\right)^n + \left(1 + \frac{1}{n}\right) \frac{n}{2} \left(1 - \left(1 - \frac{2}{n}\right)^n\right). \end{aligned}$$

Then

$$\lim_{n\to\infty} \frac{E(n)}{n} = \frac{1-e^{-2}}{2}.$$

Mock Putnam problems

A1 Let p be an odd prime. Determine the number of functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying:

- $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$;
- if $m \equiv n \pmod p$, then $f(m) = f(n)$.

First we have $f(0)^2 = f(0)$. So $f(0) = 0$ or 1. If $f(0) = 1$, we see that $f(n) = f(0) = 1$ for all n. Suppose $f(0) = 0$. Then $f(n) = 0$ when $p \mid n$. Let a be primitive mod p. Then f is determined by $f(a)$ where $f(a)^p = f(a^p) = f(a)$. There are three choices for $f(a)$. So there are 4 possible f.

A2 Let p be an odd prime. Let $g(x) = \displaystyle\sum_{k=1}^{p-1} k^{(p-1)/2} x^k \in \mathbb{F}_p[x]$. Find the largest integer n such that $(x-1)^n \mid g(x)$ in $\mathbb{F}_p[x]$.

(This is 2019 A5 but I don't agree!) We use the following formula: for $d \in \mathbb{N}$, we have

$$\sum_{k=1}^{p-1} k^d = \begin{cases} 0 & \text{if } p-1 \nmid d, \\ -1 & \text{if } p-1 \mid d. \end{cases}$$

From this, we see that $g^{(r)}(1) = 0$ for $r < (p-1)/2$. Hence the answer is $(p-1)/2$.

A4 Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1-x_n)$ for $n \geq 1$. Find

$$\lim_{n\to\infty} \frac{n(1-nx_n)}{\ln n}.$$

We have

$$\frac{1}{x_{n+1}} = \frac{1}{x_n} + \frac{1}{1-x_n} = \frac{1}{x_n} + 1 + x_n + \frac{x_n^2}{1-x_n}.$$

An easy induction shows that $x_n < 1/(n+1)$, so the series $\sum \frac{x_n^2}{1-x_n}$ converges and we have

$$\frac{1}{x_{n+1}} = n + \sum_{k=1}^{n} x_k + C_1 < n + \ln n + C_2$$

for constants $C_1, C_2 > 0$. Then from

$$x_{n+1} > \frac{1}{n + \ln n + C_2} = \frac{1}{n} - \frac{\ln n + C_2}{n(n + \ln n + C_2)} = \frac{1}{n} + O(n^{-2+\epsilon}),$$

we also have the lower bound

$$\frac{1}{x_{n+1}} > n + \ln n + C_3$$

for some constant C_3. The desired limit is 1 since

$$\lim_{n\to\infty} \frac{n\left(1 - \dfrac{n}{n + \ln n + O(1)}\right)}{\ln n} = 1.$$

A6 Let $X_1, X_2, \ldots$ be uniformly and independently chosen from $[0,1]$. Let p_n be the probability that $X_i + X_{i+1} \leq 1$ for all $i \leq n-1$. Find $\lim_{n\to\infty} p_n^{1/n}$.

The expression $p_n^{1/n}$ suggests considering the power series $\sum p_n x^n$. Let $p_0(y) = 1$ and

$$p_n(y) = \int_0^{1-y} p_{n-1}(t)dt \qquad \text{and} \qquad f(x,y) = \sum_{n=0}^{\infty} p_n(y)x^n.$$

Then

$$f(x,y) = 1 + x\int_0^{1-y} f(x,t)dt.$$

Take y-partials to find that

$$f_{yy}(x,y) = -x^2 f(x,y).$$

Solving this differential equation to find

$$f(x,0) = \frac{1+\sin x}{\cos x} = \frac{-2}{x-\pi/2} + g(x)$$

where $g(x)$ is analytic on $|x| < 3\pi/2$. Hence $p_n \asymp (2/\pi)^n$.